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Abstract

Full Text

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On the Structure of Solutions and the Correctness of Linear Systems of Differential Equations with Quasiperiodic Coefficients

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A specially constructed system of partial differential equations is investigated, and on this basis certain propositions are derived concerning linear systems of ordinary differential equations with quasiperiodic coefficients (^{1}).

1. Consider the equation

$$\frac{\partial x}{\partial u_1} + \dots + \frac{\partial x}{\partial u_m} = F(u_1, \dots, u_m)x, \tag{1}$$

where $F = \|F_{sk}(u_1, \dots, u_m)\|_1^n$ is a matrix and $x(x_1, \dots, x_n)$ is a vector. The functions F_{sk} and $\partial F_{sk}/\partial u_j$ will be assumed continuous for all values of $u_1, \dots, u_m$.

Let

$$x_{1k}(u_1, \dots, u_m), \dots, x_{nk}(u_1, \dots, u_m) \quad (k = 1, \dots, n) \tag{2}$$

be n particular solutions of equation (1).

Definition 1. We shall call the system of solutions (2) *fundamental* if it satisfies the following condition. Let $x_1(u_1, \dots, u_m), \dots, x_n(u_1, \dots, u_m)$ be any prescribed solution of equation (1); then there exist differentiable functions $A_k(u_2 - u_k, \dots, u_m - u_1)$ $(k = 1, \dots, n)$ such that

$$x_s(u_1, \dots, u_m) = A_1 x_{s1}(u_1, \dots, u_m) + \dots + A_{nx_{sn}}(u_1, \dots, u_m) \quad (s = 1, \dots, n).$$

Let $|x(u_1, \dots, u_m)|$ be the determinant of the matrix of functions (2). Then:

Theorem 1. *If* $|x(u_1, \dots, u_m)| \neq 0$ *for all values of* $u_1, \dots, u_m$, *then the system is fundamental.*

The problem of constructing equation (1) having a given system of solutions is solved in the usual way.

Note that $|x(u_1, \dots, u_m)|$ satisfies the equation

$$\frac{\partial |x|}{\partial u_1} + \dots + \frac{\partial |x|}{\partial u_m} = (F_{11} + F_{22} + \dots + F_{nn})|x|.$$

2. Let $F = A$, where $A = \|a_{sk}\|_1^n$ is a constant matrix. Then the matrix

$$x = \exp\left[\frac{\alpha_1 u_1 + \dots + \alpha_m u_m}{\alpha_1 + \dots + \alpha_m} A\right], \quad \alpha_j \ (j = 1, \dots, m) \tag{3}$$

will be a fundamental matrix of solutions of equation (1). Consequently, the study of the structure of the matrix (3) can be carried out in dependence on the matrix A, and thus the structure of the solutions of equation (1) with constant matrix A is completely determined.

3. If equation (1) is subjected to the transformation

$$x = B(u_1, \dots, u_m)Y, \tag{4}$$

where B is a nonsingular matrix, then we obtain the equation

$$\frac{\partial Y}{\partial u_1} + \dots + \frac{\partial Y}{\partial u_m} = \left[B^{-1}FB - B^{-1}\left(\frac{\partial B}{\partial u_1} + \dots + \frac{\partial B}{\partial u_m}\right)\right]Y. \tag{5}$$

Let the nonsingular matrix B, together with the matrix $\partial B/\partial u_1 + \dots + \partial B/\partial u_m$, be bounded for all positive values $u_1, \dots, u_m$ lying on the diagonal. A matrix $B(u_1, \dots, u_m)$ having this property will be called an A. M. Lyapunov matrix.

We shall call a function $f(u_2 - u_1, \dots, u_m - u_1)$ a function constant on the diagonal. Similarly, we shall call a matrix constant on the diagonal if its elements are functions constant on the diagonal.

If $B(u_1, \dots, u_m)$ in the transformation (4) is a matrix constant on the diagonal or constant everywhere, then equation (5) assumes the simplest form

$$\frac{\partial Y}{\partial u_1} + \dots + \frac{\partial Y}{\partial u_m} = B^{-1}FBY.$$

Definition 2. Equation (1) will be called **reducible** if, by means of transformation (4) with a Lyapunov matrix B, it is transformed into an equation with a matrix constant on the diagonal or constant everywhere.

Let us note that if in equation (1) the matrix F is constant on the diagonal, then the matrix

$$x = \exp\left[\frac{\alpha_1 u_1 + \dots + \alpha_m u_m}{\alpha_1 + \dots + \alpha_m} F\right], \qquad \alpha_j = \text{const},$$

will be a solution of this equation.

It is easy to obtain extensions of N. P. Erugin' s theorem (^{2}).

Theorem 2. *In order that equation (1) be reducible, it is necessary and sufficient that the fundamental system of solutions of this equation can be represented in the form*

$$x(u_1, \dots, u_m) = B(u_1, \dots, u) \exp \left[\frac{\alpha_1 u_1 + \dots + \alpha'_m u_m}{\alpha_1 + \dots + \alpha_m} A \right],$$

where B is a Lyapunov matrix, $\alpha_k = \text{const}$, and A is a matrix constant on the diagonal or constant everywhere.

Theorem 3. *If the coefficients F_{sk} in equation (1) are periodic functions with one and the same real period ω in all variables u_k, then equation (1) is reducible by means of a periodic matrix.*

If equation (1) is considered on the diagonal $u_k = t$, then this theorem gives the well-known theorem of A. M. Lyapunov on the reducibility of linear systems of equations with periodic (with a common period) coefficients.

4. Let the matrix F in equation (1) satisfy the condition

$$F(u_1 + \omega_1, \dots, u_m + \omega_m) = F(u_1, \dots, u_m).$$

Along the diagonal, equation (1) gives the equation

$$\frac{dx}{dt} = P(t)x \tag{6}$$

with the quasiperiodic (^{3}) matrix $P(t) = \|F_{sk}(t, \dots, t)\|_1^n$. Let $x(u_1, \dots, u_m)$ be a fundamental matrix of solutions of equation (1). Then $x(u_1 + \omega_1, \dots, u_m + \omega_m)$, by virtue of the periodicity of the matrix F, is also a solution of equation (1), and on the basis of the preceding

$$x(u_1 + \omega_1, \dots, u_m + \omega_m) = x(u_1, \dots, u_m)C, \tag{7}$$

where C is a matrix constant on the diagonal or constant everywhere.

Thus, we arrive at a relation analogous to the one that enabled Floquet (^{2}) to establish the structure of the solutions of systems of ordinary equations with periodic coefficients.

Theorem 4. *If equation (1) is such that the matrix C in relation (7) is everywhere constant, then this equation is reducible to an equation with an everywhere constant matrix by means of a periodic matrix.*

The converse theorem also holds.

Theorem 5. *If equation (1) is reducible to an equation with an everywhere constant matrix by means of a periodic matrix, then the matrix C in relation (7) is everywhere constant.*

Applying the result obtained to equation (6), we have:

In order that equation (6) be reducible by means of a quasiperiodic matrix, it is necessary and sufficient that in the relation

$$x(t+\omega_1,\dots,t+\omega_m) = x(t,\dots,t)D$$

the constant matrix D should not depend on the periods ω_k.

The structure of the solutions of equation (1), and hence also of equation (6), is completely determined by the matrix C.

1°. C is everywhere constant. Then

$$x(u_1,\dots,u_m) = B(u_1,\dots,u_m)\exp\left[\frac{\alpha_1 u_1+\dots+\alpha_m u_m}{\alpha_1+\dots+\alpha_m}A\right],$$

where B is periodic, $\alpha_j = \text{const}$, and A is a constant matrix. For equation (6) this gives

$$x(t) = \Phi(t)e^{tA}, \tag{8}$$

where $\Phi(t)$ is a quasiperiodic matrix; i.e., we obtain a result analogous to Floquet' s results.

2°. $C = [\lambda_1,\dots,\lambda_n]$, where λ_k are constant diagonal functions. In this case one obtains the relations

$$x_{jk}(u_1+\omega_1,\dots,u_m+\omega_m) = \lambda_j x_{jk}(u_1,\dots,u_m),$$

which are satisfied by functions of the form

$$x_{jk}(u_1,\dots,u_m) = \Phi_{jk}(u_1,\dots,u_m)e^{R_j(u_1,\dots,u_m)},$$

where Φ_{jk} are periodic in the variables u_i with periods ω_i,

$$\frac{\partial R_j}{\partial u_1}+\dots+\frac{\partial R_j}{\partial u_m} = K_j(u_1,\dots,u_m),$$

and K_j is periodic in the variables u_i with periods ω_i. In particular,

$$R_j = \int_0^{u_1} K_j(z, u_2 - u_1 + z, \dots, u_m - u_1 + z)\, dz.$$

For equation (6) this gives

$$x_{jk}(t) = \varphi_{jk}(t) e^{m_j(t)}, \qquad x_{jk}(t) = \varphi_{jk}(t) \exp\left[\int_0^t l_j(z)\, dz\right],$$

where the functions $\varphi_{jk}(t) = \Phi_{jk}(t, \dots, t)$, $l_j(t) = K_j(t, \dots, t)$, and $m_j'(t)$ are quasiperiodic.

If

$$\int_0^t l_j(z)\, dz$$

are quasiperiodic functions, then we again obtain form (8), while if these integrals are not quasiperiodic, then a form of solutions of equation (6) is obtained which no longer agrees with Floquet theory. In this latter case equation (6) is not reducible, but if the characteristic numbers of this equation and of the equation adjoint to it are denoted respectively by α_k and μ_k, then $\alpha_k + \mu_k = 0$, i.e. equation (6) is regular.

3°. If the matrix C has Jordan form, then the structure of the solutions of equation (1) is also established. Let, for example,

$$C = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where λ is a function constant on the diagonal. Then

$$x_{k1}(u_1, \dots, u_m) = \Phi_{k1}(u_1, \dots, u_m) e^{R(u_1, \dots, u_m)},$$

$$x_{k2}(u_1, \dots, u_m) =$$

$$= \left[\Phi_{k2}(u_1, \dots, u_m) + \gamma(u_1, \dots, u_m)\Phi_{k1}(u_1, \dots, u_m)\right] e^{R(u_1, \dots, u_m)} \qquad (k = 1, 2),$$

where Φ_{k1} and Φ_{k2} are periodic in the variables u_k with periods ω_k, and R and γ are such that the sums of their partial derivatives are periodic functions. In this case equation (6) is again regular.

Thus, a broad class of regular, but in the general case nonreducible, equations has been given.

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