MATHEMATICS

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Abstract

Full Text

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ON LOCALLY NORMAL GROUPS

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Subgroups of direct products of finite groups are locally normal. However, it is known that the locally normal groups which are subgroups of such products are not exhausted by them. Therefore the question naturally arises of conditions under which a locally normal group is isomorphic to a subgroup of some direct product of finite groups (see (^{1})).

1. The following classes of groups are known to be isomorphically embeddable in direct products of finite groups:

1) the class of locally normal groups with finite Sylow p-subgroups for all prime numbers p (^{2});

2) the class of countable finitely approximable locally normal groups (^{3});

3) the class of locally normal groups without center (^{4}).

A direct generalization of the results 2)–3) is the following

Theorem 1. *A finitely approximable locally normal group with countable center is isomorphic to a subgroup of some direct product of finite groups.*

Remark. If the center of a locally normal finitely approximable group is uncountable, then it is not necessarily isomorphic to a subgroup of a direct product of finite groups. This can be verified from the example of a finitely approximable abelian p-group not decomposable into a direct product of cyclic groups (see (^{5}), p. 168).

Theorem 1 follows immediately from the following lemma and its corollaries.

Lemma 1. *Let $\mathfrak{G}$ be a locally normal subgroup of a full direct product of a countable number of finite groups.*

Then, if $\mathfrak{K}$ is a countable subgroup of the group $\mathfrak{G}$, there exists an isomorphic mapping of the group $\mathfrak{G}$ into some full direct product of a countable set of finite groups under which the image of each element of the group $\mathfrak{K}$ has only a finite number of components distinct from the identity.

Proof. Since the group $\mathfrak{G}$ is isomorphic to a subdirect product of a countable number of finite groups, it has a descending invariant series with finite factors, whose terms are indexed by natural numbers. As a locally normal group, the group $\mathfrak{G}$ has a countable normal divisor $\mathfrak{H}$ containing the countable subgroup $\mathfrak{K}$. It is therefore sufficient to prove the lemma for the group $\mathfrak{H}$.

In view of the countability of $\mathfrak{H}$, its elements can be numbered by natural numbers:

$$H_1, H_2, \dots, H_n, \dots$$

Since the group $\mathfrak{G}$ is locally normal, one can construct a chain of its normal divisors $\mathfrak{H}_n$ $(n = 1, 2, \dots)$, contained in $\mathfrak{H}$, satisfying the conditions:

1) $\mathfrak{H}_n \subseteq \mathfrak{H}_m$ for $n < m$;
2) $\mathfrak{H}_n$ contains the elements H_i $(i = 1, 2, \dots, n)$.

As already noted, the group $\mathfrak{G}$ has a descending invariant series with finite factors

$$\mathfrak{G} \supset \mathfrak{G}^{(1)} \supset \cdots \supset \mathfrak{G}^{(n)} \supset \cdots \supset E.$$

Therefore, in view of the finite approximability of the group $\mathfrak{G}$, there exists a countable chain

$$\mathfrak{S}_1 \supseteq \mathfrak{S}_2 \supset \cdots \supseteq \mathfrak{S}_n \supseteq \cdots \supseteq E$$

of invariant subgroups of finite index in $\mathfrak{G}$ such that $\mathfrak{S}_n \cap \mathfrak{H}_n = 1$. Consider the chain of subgroups $\mathfrak{P}_n = \mathfrak{S}_n \times \mathfrak{H}_n$ $(n = 1, 2, \dots)$. By a direct count one can show that $\bigcap_n \mathfrak{P}_n = 1$. But the latter means the validity of the assertion of the lemma for the subgroup $\mathfrak{H}$, and hence also for the subgroup $\mathfrak{K}$.

Corollary 1. A countable finitely approximable locally normal group is isomorphic to a subgroup of some direct product of finite groups (^{3}).

Corollary 2. Let $\mathfrak{G}$ be a locally normal subgroup of the full direct product of a countable set of finite groups. Then the group $\mathfrak{G}$ is isomorphic to some subdirect product of finite groups $\mathfrak{G}_n$ $(n = 1, 2, \dots)$, each element of which has only a finite number of components not belonging to the centers of the groups $\mathfrak{G}_n$ $(n = 1, 2, \dots)$.

Corollary 3. A locally normal subgroup of the full direct product of a countable set of finite groups decomposes into the product of its center and a subgroup isomorphically embeddable in a direct product of finite groups.

Corollary 4. A locally normal subgroup with countable center of the full direct product of a countable set of finite groups is isomorphically embeddable in the direct product of a countable set of finite groups.

Corollary 5. A locally normal finitely approximable group is isomorphic to a subdirect product of some countable finitely approximable group and a group without center.

2. As already noted, locally normal groups are not exhausted by direct products of finite groups and subgroups of such products. Therefore it is natural to broaden somewhat the class of direct products into which locally normal groups are isomorphically embeddable.

A partial answer to this question is given by

Theorem 2. *A locally normal group with abelian Sylow p-subgroups (for all p) is isomorphic to a subgroup of the direct product of some abelian group and finite groups without center.*

Corollary. A locally normal group all of whose cyclic subgroups are complemented is isomorphic to a subgroup of the direct product of finite completely factorizable groups ([6]).

If the Sylow p-subgroups of a locally normal group are nonabelian, then an analogous theorem is false; namely, one cannot assert that a locally normal group (even a finitely approximable one) is a subdirect product of a ZA-group and a group without center. This can be verified from the following example.

Example. Let A and B be generators of the quaternion group, and let an element C ($C^3 = 1$) act on them as follows:

$$C^{-1}AC = B, \qquad C^{-1}BC = A^{-1}B.$$

Then the elements A, B, C generate a group $\mathfrak{H}$ of order 24 with center of order 2. The direct product of the groups $\mathfrak{H}_n$ ($n = 1, 2, ...$), isomorphic to the group $\mathfrak{H}$, is a locally normal group which cannot be represented as a subdirect product of a locally normal ZA-group and a locally normal group without center.

Theorem 2 is obtained from the following propositions on locally finite groups with abelian Sylow p-subgroups by applying the results of [4].

Lemma 2. *If a finite group with abelian Sylow p-subgroups (p fixed) has at least one element of p-order belonging to its center, then it has a normal divisor whose factor group is an abelian p-group.*

Lemma 2 is obtained by applying Lemma 14.4.1 from [7] (see p. 206, the Transfer mapping).

Lemma 3. *If a locally finite group $\mathfrak{G}$ with abelian Sylow p-subgroups (p fixed) has at least one element of p-order belonging to its center, then it has a normal divisor whose factor group is an abelian p-group.*

Proof. Let Z be an arbitrary element of p-order belonging to the center of the group $\mathfrak{G}$, and let $[\mathfrak{G}_\alpha]$ be a local system of finite subgroups of the group $\mathfrak{G}$ containing Z. Let $\mathfrak{N}_\alpha$ be the subgroup of the group $\mathfrak{G}_\alpha$ generated by all elements of orders relatively prime to the number p. By Lemma 2, the group $\mathfrak{G}_\alpha/\mathfrak{N}_\alpha$ is distinct from the identity. Since $\mathfrak{G}_\alpha/\mathfrak{N}_\alpha$ is a p-group and the Sylow p-subgroups of the group $\mathfrak{G}_\alpha$ are abelian, the group $\mathfrak{G}_\alpha/\mathfrak{N}_\alpha$ is also abelian. It is clear that from the relation $\mathfrak{G}_\alpha \subseteq \mathfrak{G}_\beta$ there follows the relation $\mathfrak{N}_\alpha \subseteq \mathfrak{N}_\beta$. Thus, the groups $\mathfrak{N}_\alpha$ form a local system $[\mathfrak{N}_\alpha]$. Put

$$\mathfrak{N} = \bigcup_\alpha \mathfrak{N}_\alpha.$$

It is not hard to verify that $\mathfrak{N}$ is such a normal divisor of the group $\mathfrak{G}$, not containing the element Z, whose factor group is an abelian p-group.

From Lemma 3 it follows:

Theorem 3. *A locally finite group $\mathfrak{G}$ with abelian Sylow p-subgroups (for all p) is the direct product of an abelian group and a group without center.*

Proof. By Lemma 3, in the group $\mathfrak{G}$ there exists a normal divisor $\mathfrak{N}$ not containing elements of the center $\mathfrak{Z}$ of the group $\mathfrak{G}$, whose factor group is abelian. Thus, $\mathfrak{N} \cap \mathfrak{Z} = 1$.

Let $\mathfrak{S}$ be the union of all terms of the upper central series of the group $\mathfrak{G}$. Then, if $\mathfrak{N} \cap \mathfrak{S} \neq 1$, also $\mathfrak{N} \cap \mathfrak{Z} \neq 1$ (see [8]). The contradiction obtained shows that $\mathfrak{N} \cap \mathfrak{S} = 1$. Since $\mathfrak{G}/\mathfrak{S}$ has no center, and $\mathfrak{G}/\mathfrak{N}$ is abelian, the theorem is proved.

Corollary. If the Sylow p-subgroups (for all p) of a locally finite group $\mathfrak{G}$ are abelian, then the factor group $\mathfrak{G}/\mathfrak{Z}$ of the group $\mathfrak{G}$ by its center $\mathfrak{Z}$ has no center.

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