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Abstract

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MATHEMATICS

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THE FOURIER METHOD FOR DIFFERENTIAL EQUATIONS CONTAINING A SECOND DERIVATIVE WITH RESPECT TO TIME

(Presented by Academician I. G. Petrovskii on May 18, 1962)

1. Let, in a bounded domain Ω of the space $x = (x_1, \dots, x_n)$, there be given a formally self-adjoint differential operator

$$Au = \sum_{|\alpha|=|\beta|\leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u), \quad (1,1)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha = \partial^\alpha / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$, and $a_{\alpha\beta}(x)$ are real functions in Ω , symmetric with respect to permutations of the indices. Suppose that there exists a constant $C_0 > 0$ and that for every real n -dimensional vector ξ , for all $x \in \bar{\Omega}$, the inequality

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta \geq C_0 |\xi|^{2m}, \quad (1,2)$$

holds, where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ and $|\xi|$ is the Euclidean norm of ξ .

Consider in the cylinder $Q = \Omega \times [0, T]$ the equation

$$Au + \frac{\partial^2 u}{\partial t^2} = f(x, t). \quad (1,3)$$

For equation (1,3) we pose the following mixed problem: to determine in Q a solution $u(x, t)$ of this equation satisfying the conditions

$$u|_{t=0} = \varphi(x), \quad \frac{\partial u}{\partial t} \Big|_{t=0} = \psi(x); \quad (1,4)$$

$$\left. \frac{\partial^k u}{\partial \nu^k} \right|_{\Gamma} = 0 \quad (k = 0, 1, \dots, m-1), \quad (1,5)$$

where $\Gamma = S \times [0, T]$ is the lateral surface of Q ; ν is the outward normal to Γ .

We shall prove the solvability of problem (1,3)–(1,5) by the Fourier method. For this purpose denote by $\{v_i\}$ a complete in $L_2(\Omega)$ system of orthonormal eigenfunctions of the equation

$$Av = \lambda v \quad (1,6)$$

with boundary conditions

$$\left. \frac{\partial^k v}{\partial \nu^k} \right|_S = 0 \quad (k = 0, 1, \dots, m-1), \quad (1,7)$$

corresponding to the system of eigenvalues $\{\lambda_i\}$, numbered in increasing order. Applying formally the method of separation of variables, we obtain the solution of problem (1,3)–(1,5) in the form:

$$\begin{aligned} u(x, t) = & \sum_{i=1}^{\infty} v_i(x) \left(\varphi_i \cos \sqrt{\lambda_i} t + \frac{\psi_i}{\sqrt{\lambda_i}} \sin \sqrt{\lambda_i} t \right) \\ & + \sum_{i=1}^{\infty} \frac{v_i(x)}{\sqrt{\lambda_i}} \int_0^t f_i(\tau) \sin \sqrt{\lambda_i}(t - \tau) d\tau, \end{aligned} \quad (1,8)$$

where φ_i , ψ_i and f_i are the Fourier coefficients of the functions φ , ψ and f with respect to the system $\{v_i\}$. To justify the Fourier method it is necessary to determine the conditions under which the series (1.8) defines an actual solution of the mixed problem.

Numerous investigations ^(1–6) have been devoted to the question of justifying the Fourier method. In the present paper this method is justified for the mixed problem (1.3)–(1.5) from the point of view of solutions in the Sobolev spaces $W_{x,t,2}^{m,l}(Q)$ (l a natural number) ⁽⁸⁾, under sufficiently weak assumptions on the smoothness of the coefficients of equation (1.3). We mainly use the method of O. A. Ladyzhenskaya ⁽¹⁾.

2. We proceed to the statement of the main results. Denote by

$\overset{\circ}{W}_2^m(\Omega)$ the closure, in the metric of $W_2^m(\Omega)$ ⁽⁷⁾, of the set of all infinitely differentiable functions with compact supports in Ω . Similarly, denote by $\overset{\circ}{W}_2^{m,1}(Q)$ the closure, in the norm $W_{x,t,2}^{m,1}(Q)$, of the set of all infinitely differentiable functions in the cylinder Q which are equal to zero near Γ . Let

$$D(u, v) = \int_{\Omega} \sum_{|\alpha|=|\beta| \leq m} a_{\alpha\beta} D^{\alpha} u D^{\beta} v dx. \quad (2.1)$$

We shall assume that, for $u \in \overset{\circ}{W}_2^m(\Omega)$, the condition

$$D(u) \geq C_1 \|u\|_0^2,$$

holds, where $C_1 > 0$.

A function $v \in \overset{\circ}{W}_2^m(\Omega)$ will be called a generalized eigenfunction of the problem (1.6), (1.7) if it is not identically equal to zero and

$$D(v, \xi) = \lambda(v, \xi) \quad (2.2)$$

for any function $\xi \in \overset{\circ}{W}_2^m(\Omega)$. The number λ appearing in (2.2) is called the eigenvalue corresponding to the function v .

Under the assumptions that Ω is an arbitrary bounded domain and $a_{\alpha\beta} \in L_p(\Omega)$, where $p > n/(m - |\alpha|)$ and $p \geq 2$, the question of the existence of a complete, orthonormal in $L_2(\Omega)$ and in $\overset{\circ}{W}_2^m(\Omega)$, system of generalized eigenfunctions of the problem (1.6), (1.7) is solved by a known variational method ⁽¹⁰⁾.

In the series (1.8), by $\{v_i\}$ we shall mean the system of generalized eigenfunctions of the problem (1.6), (1.7). In what follows, by $\|u\|_k$ and $\|u\|_{k,l}$ we shall denote the norms in $W_2^k(\Omega)$, respectively in $W_{x,t,2}^{k,l}(Q)$. In addition, let us assume that S belongs to the class R^l (l a natural number) ⁽⁸⁾. To this class belongs, for example, any closed surface that is continuously differentiable $l + 1$ times.

Theorem. *Let the following conditions be satisfied: 1) Ω is a finite domain bounded by a surface of class R^{km} , where $k \geq 2$; 2) the coefficients $a_{\alpha\beta} \in W_p^{|\alpha|+m(k-2)}(\Omega)$, where $p > n/(km - |\alpha|)$ and $p \geq 2$; 3) $\varphi \in W_2^{km}(\Omega)$, $\psi \in W_2^{m(k-1)}(\Omega)$, and, moreover, the functions $\varphi, A\varphi, \dots, A^{[\frac{k-1}{2}]} \varphi$ and the functions $\psi, A\psi, \dots, A^{[\frac{k-2}{2}]} \psi$ belong to $\overset{\circ}{W}_2^m(\Omega)$; 4) $f \in W_{x,t,2}^{m(k-1), k-1}(Q)$, and, moreover, the functions $f, Af, \dots, A^{[\frac{k-2}{2}]} f$ belong to $\overset{\circ}{W}_2^{m,1}(Q)$.*

Then the series (1.8) converges in the norm of $W_2^{km}(\Omega)$ uniformly with respect to $t \in [0, T]$, and the series obtained by termwise differentiating it l times with respect to t ($l \leq k$) converges in the norm of $W_2^{m(k-l)}(\Omega)$ uniformly in $t \in [0, T]$. Moreover, the sum $u(x, t)$ of the series (1.8) is a solution of the problem (1.3) – (1.5) in $W_{x,t,2}^{km,k}(Q)$.

and for all $t \in [0, T]$ the inequality

$$\sum_{l=0}^k \left\| \frac{\partial^l u}{\partial t^l} \right\|_{m(k-l)}^2 \leq C_2 (\|\varphi\|_{km}^2 + \|\psi\|_{m(k-1)}^2 + \|f\|_{m(k-1), k-1}^2), \quad (2,3)$$

holds, where the constant C_2 depends on T , the domain Ω , and $a_{\alpha\beta}$.

We note that the theorem remains valid also for $k = 1$. In this case Ω may be any bounded domain, and the coefficients $a_{\alpha\beta} \in L_p(\Omega)$, where $p > n/(m - |\alpha|)$ and $p \geq 2$. The sum $u(x, t)$ of the series (1.8) is a generalized solution of the mixed problem (1.3)–(1.5) in the sense of Ladyzhenskaya ⁽¹⁾, i.e. $u \in \dot{W}_2^{m,1}(Q)$, assumes the value φ at $t = 0$ in the norm $L_2(\Omega)$, and satisfies the identity

$$\int_0^T \left\{ \left(\frac{\partial u}{\partial t}, \frac{\partial \Phi}{\partial t} \right) - D(u, \Phi) + (f, \Phi) \right\} dt + (\psi, \Phi)|_{t=0} = 0. \quad (2,4)$$

for any function $\Phi \in \dot{W}_2^{m,1}(Q)$ equal to zero for $t = T$. The uniqueness of the generalized solution is proved analogously to ⁽¹⁾.

The proof of the theorem is based on a priori estimates in the norms $W_2^l(\Omega)$ of solutions of elliptic equations, which were obtained by various authors and, under the weakest restrictions, were proved in ⁽⁹⁾.

Namely, for any function $u \in W_2^l(\Omega) \cap \dot{W}_2^m(\Omega)$ ($l \geq 2m$), in the case $S \in R^l$, the inequality

$$\|u\|_l \leq C(\|Au\|_{l-2m} + \|u\|_0) \quad (2,5)$$

holds. This inequality was proved in ⁽⁹⁾ under the assumption of sufficient smoothness of the coefficients of the operator A . It can be shown that it remains valid if $a_{\alpha\beta} \in W_p^{|\alpha|+l-2m}(\Omega)$, where $p > n/(l - |\alpha|)$ and $p \geq 2$. Hence it follows that the generalized eigenfunctions of problem (1.6), (1.7) belong to $W_2^l(\Omega)$. In addition, from (2.5) it follows directly that

Lemma 1. If conditions 1) and 2) of the theorem are satisfied, then for any function φ satisfying condition 3) of this theorem the inequality

$$\|\varphi\|_{km}^2 \leq \begin{cases} C_3 \|A^{\frac{k}{2}} \varphi\|_0^2, & k \text{ even,} \\ C_3 D(A^{\frac{k-1}{2}} \varphi), & k \text{ odd.} \end{cases} \quad (2,6)$$

holds.

The following Lemma 2 is a consequence of the definition of the generalized eigenfunctions of problem (1.6), (1.7) and of the completeness of this system in $L_2(\Omega)$ and in $\dot{W}_2^m(\Omega)$.

Lemma 2. If the conditions of Lemma 1 are satisfied, then the following Steklov-type equality holds:

$$\sum_{i=1}^{\infty} \lambda_i^k \varphi_i^2 = \begin{cases} \|A^{\frac{k}{2}} \varphi\|_0^2, & k \text{ even,} \\ D(A^{\frac{k-1}{2}} \varphi), & k \text{ odd.} \end{cases} \quad (2,7)$$

Moreover, the following also holds.

Lemma 3. If conditions 1), 2) and 4) of the theorem are satisfied, then

$$\sum_{i=1}^{\infty} \lambda_i^{k-s-1} |f_i^{(s-1)}(t)|^2 \leq C_4 \|f\|_{m(k-1), k-1}^2 \quad (s = 1, \dots, k-1). \quad (2,8)$$

Here the series (2.8) converge uniformly on $[0, T]$.

The theorem is proved by direct application of Lemmas 1-3. From the theorem, with the aid of the embedding theorems of S. L. Sobolev ⁷, one obtains, as a special case, the existence of a solution of the mixed problem (1,3)–(1,5) in the spaces $C_{x,t}^{mr,r}(\bar{Q})$.

Corollary. If the conditions of the theorem are fulfilled for $k = [n/2m] + r + 1$, then the series (1,8) and the series obtained from it by termwise differentiation $D^{m(r-s)} \partial^s / \partial t^s$ for $s = 0, 1, \dots, r$, converge uniformly with respect to (x, t) in \bar{Q} . For $k = [n/2m] + 3$, the sum of the series (1,8) is a classical solution (a solution in $C_{x,t}^{2m,2}(\bar{Q})$) of the mixed problem (1,3)–(1,5).

Thus, the results of O. A. Ladyzhenskaya ¹ on the justification of the Fourier method carry over to equations of higher orders of the form (1,3).

The existence of a classical solution of the mixed problem (1,3)–(1,5) has been proved by us under high smoothness requirements on the boundary surface ($S \in R^{2m+[n/2]+1}$). At the same time, the assumptions concerning the smoothness of the coefficients are relatively weak, namely: $a_{\alpha\beta} \in W_2^{|\alpha|+[n/2]+1}(\Omega)$. As V. A. Il'in ² showed, in the case $m = 1$ such stringent restrictions on the smoothness of S can be reduced to minimal ones if, in the definition of a classical solution, one gives up continuity of the second derivatives in the closed cylinder; however, for $m \geq 2$ this question has not yet been resolved.

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