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Abstract

Full Text

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ABSTRACT SET FUNCTIONS AND S. L. SOBOLEV EMBEDDING THEOREMS

(Presented by Academician S. L. Sobolev on 23 I 1962)

In this paper we consider additive abstract set functions $\varphi(E)$, defined for all L-measurable sets E from some n-dimensional domain $\Omega \in R_n$, with values in a B-space X. A detailed study of such functions was carried out in ([1,7]). Some results are also contained in ([3,4]).

Most of the concepts and notation that we use were introduced in ([1,3]). Only occasionally, instead of Ψ_p or Φ_p, following V. B. Korotkov, shall we write $\Psi_p(\Omega)$ or $\Psi_p(X, \Omega)$ (similarly $\Phi_p(\Omega)$ and $\Phi_p(X, \Omega)$).

We shall consider abstract additive functions belonging to the B-space $\Phi_1(\Omega)$ with norm $\sup\limits_{E_1 \cap E_2 = 0} \|\varphi(E_1) - \varphi(E_2)\|_X$, or with the equivalent norm $\sup\limits_{E \subset \Omega} \|\varphi(E)\|_X$, as well as functions from the B-spaces $\overline{\Phi}_1(\Omega)$, $\overline{\overline{\Phi}}_1(\Omega)$, considered in ([3]). Here $\overline{\overline{\Phi}}_1$ is the space of absolutely continuous functions, i.e., such functions for which $\|\varphi(E)\|_X \to 0$ as $mE \to 0$, and $\overline{\Phi}_1$ is the space of additive and normal functions, i.e., additive functions tending to 0 in the norm of X on a vanishing sequence of sets.

It is easy to establish ([3]) that $\overline{\Phi}_1$ coincides with the space of countably additive abstract functions. In addition, we shall consider the space $\Phi_p(\Omega)$, the space of additive functions with bounded norm

$$\|\varphi\|_{\Phi_p} = \sup_{\omega \in L_{p'}} \frac{\left\| \int_\Omega \omega \, d\varphi(E) \right\|_X}{\|\omega\|_{L_{p'}}}, \qquad p > 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

It can be shown ([1,3]) that

$$\Phi_p \subset \overline{\overline{\Phi}}_1 \subset \overline{\Phi}_1 \subset \Phi_1.$$

Let $\varphi(E) \in \overline{\Phi}_1(X, \Omega)$.

Theorem 1. *If $\varphi(E)$ is normal, then $\psi_\varphi(E') = \varphi(E' \cap E)$ is also normal as an abstract function of sets E' with values in $\overline{\Phi}_1(X, \Omega)$.*

In other words, if $\varphi(E) \in \overline{\Phi}_1$ and E_k is a vanishing sequence, then

$$\sup_{E\in\Omega} \|\varphi(E \cdot E_k)\|_X = \|\varphi\|_{\Phi_1(E_k)} \to 0 \quad \text{as } k \to \infty.$$

The theorem follows easily from the equality

$$E \cap \left[\bigcup_i E_i'\right] = \bigcup_i [E \cap E_i'],$$

where the E_i' are disjoint, and from theorem 3 ([3]).

From theorem 1, in particular, there follows a very simple proof of the following theorem (see ([3])).

Theorem 2. *If $\varphi(E) \in \overline{\Phi}_1$, $E_{k+1} \subset E_k$, and $E_0 = \lim\limits_{k\to\infty} E_k$, while $\omega(x)$ is a bounded numerical function such that $\omega(x) = a_0$ for $x \in E_0$, then*

$$\lim_{k\to\infty} \int_{E_k} \omega(x)\, d\varphi(E) = a_0 \varphi(E_0).$$

Indeed:

$$\left\| \int_{E_k} \omega(x)\, d\varphi(E) - a_0\varphi(E_0) \right\|_X =$$

$$= \left\| \int_{E_k - E_0} \omega(x)\, d\varphi(E) \right\|_X \le \max_{x\in\Omega} |\omega(x)|\, \|\varphi\|_{\Phi_1(E_k - E_0)} \le \varepsilon$$

for $k > K(\varepsilon)$, since $\{E_k - E_0\}$ is a vanishing sequence of sets.

Theorem 3. If $E_k \in \Omega$ is an increasing (decreasing) sequence of sets, $E_0 = \lim_{k\to\infty} E_k$, and $\varphi(E) \in \overline{\Phi}_1$, then

$$\lim_{k\to\infty} \|\varphi(E_k) - \varphi(E_0)\|_X = 0,$$

$$\lim_{k\to\infty} \|\varphi\|_{\Phi_1(E_k)} = \|\varphi\|_{\Phi_1(E_0)}.$$

Theorem 4. Let E_n be any sequence of sets from Ω. Denote

$$E_0^s = \limsup_n E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \quad \text{and} \quad E_0^i = \liminf_n E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Then, if $\varphi(E) \in \overline{\Phi}_1$, then

$$\|\varphi\|_{\Phi_1(E_0^s)} \geq \overline{\lim_n} \|\varphi\|_{\Phi_1(E_n)},$$

$$\|\varphi\|_{\Phi_1(E_0^i)} \leq \underline{\lim_n} \|\varphi\|_{\Phi_1(E_n)}.$$

From Theorem 4 it follows:

Theorem 5. In order that a normal function be absolutely continuous, it is necessary and sufficient that it vanish on every set of measure zero.

This theorem generalizes the well-known theorem for numerical normal set functions (see, for example, ([5]), p. 53).

Let now $\varphi(E) \in \Phi_p$, $p > 1$, or $\in \overline{\Phi}_1$ for $p = 1$.

Lemma 1. If $\varphi(E) \in \Phi_p$, $p > 1$ ($\varphi(E) \in \overline{\Phi}_1$ for $p = 1$), $\omega_1(x) \in L(\Omega)$, $\omega_2(x) \in L_{p'}(\Omega)$, $\omega_1 = \omega_2 = 0$ outside Ω, $\frac{1}{p} + \frac{1}{p'} = 1$, then in the triple convolution

$$\psi(y) = \int \omega_1(z) \left[\int \omega_2(y - z - x)\, d_x \varphi(E) \right] dz \tag{1}$$

the associativity law is valid and the order of performing the operations may be changed. The formula holds

$$\psi(y) = \int \left[\int \omega_1(z) \omega_2(y - z - x)\, dz \right] d_x \varphi(E).$$

This lemma for bounded $\omega_1(x)$ and $\omega_2(x)$ and for $\varphi(E) \in \Psi_p(\Omega)$, $p > 1$, was proved in (1) (Ψ_p is the space of functions from Φ_p continuous with respect to translation in the norm Φ_p).

From Lemma 1 and Lemmas III and IV of (1), pp. 314-315, follows the equality

$$[z_\varphi(y)]_h = z_{\varphi_h}(y),$$

where $z_\varphi(y) = \varphi(E + y)$.

Hence it is easily obtained:

Theorem 6. If $\varphi(E) \in \Phi_p$ ($\overline{\Phi}_1$ for $p = 1$) and $z_\varphi(y)$ is measurable as a function of the point y with values in Φ_p ($\overline{\Phi}_1$), then $\varphi(E)$ is continuous with respect to translation in the norm Φ_p ($\overline{\Phi}_1$).

A certain modification of Lemma 1 is

Lemma 2. If $\varphi(E) \in \overline{\Phi}_1$ and is continuous with respect to translation in the norm Φ_1, and $\omega_1(x)$ and $\omega_2(x)$ are bounded functions, then in the triple convolution (1) the associativity law is also valid.

Hence it is also easy to obtain the equality $z_{\varphi_h}(y) = [z_\varphi(y)]_h$, whence it follows:

Theorem 7. The mean functions $\varphi_h(E)$ are dense in the space $\Psi_1(\Omega)$, where $\Psi_1(\Omega)$ is the space of normal functions continuous under translation in the metric $\Phi_1(\Omega)$.

From this it follows immediately:

Theorem 8. If a function $\varphi(E) \in \Psi_1(\Omega)$, then it is absolutely continuous in the metric $\Phi_1(\Omega)$.

This theorem generalizes a similar theorem for numerical functions (see, for example, ([5]), p. 140).

Let us now consider the space $\Phi_p^l(\Omega)$, $\mathbf{p} = (p_1, p_2, \dots, p_n)$, $p_i \geqslant 1$, $l = (l_1, l_2, \dots, l_n)$, $l_i \geqslant 1$. To the space $\Phi_p^l(\Omega)$ we shall assign all normal functions $\varphi(E)$ whose generalized derivatives (see (1)) $\psi^{l_i}(E)$ of order l_i in the direction x_i belong to $\Phi_{p_i}(\Omega)$ $(\overline{\Phi}_1(\Omega),\ p_i = 1)$.

Theorem 9. If a function $\varphi(E)$ belongs to $\Phi_p^l(\Omega)$, then it is constant under translation on sets $E \in \Omega$ whose measure is zero.

From this it follows:

Theorem 10. A function $\varphi(E) \in \Phi_p^l$ is equal to zero on a set of zero measure u, and consequently is absolutely continuous.

Obviously, $\Psi_p^l \subset \Phi_p^l(\Omega)$, where Ψ_p^l is the space of normal functions having all generalized derivatives $\Psi^1(E) \in \Phi_p$, $|1| = l$. From Theorem 10 it follows that for the derivative function $\varphi(E) \in \Phi_p^l$ the equality

$$\psi^1(E) = \lim_{|\Delta x_i| \to 0} \frac{\Delta_{x_1}^{l_1} \cdots \Delta_{x_n}^{l_n} \varphi(E)}{|\overrightarrow{\Delta x_i}|^l}, \qquad l = |\vec{1}| = l_1 + l_2 + \dots + l_n,$$

holds, where $\Delta_{x_1}^{l_1} \cdots \Delta_{x_n}^{l_n} \varphi(E)$ denotes the corresponding l-th difference of the function $\varphi(E)$, and the limit is understood in the sense of the metric $\Phi_1(\Omega)$.

The last equality can also be rewritten in the form

$$z_{\psi^1}(x) = \frac{\partial^l z_\varphi(x)}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}},$$

where $z_{\psi^1}(x)$ and $z_\varphi(x)$ are abstract functions of points with values in $\Phi_1(X, \Omega)$. Hence, for $\Psi^1(E) \in \Psi_p$ for all $|1| = l$, Theorem 6 ([3]) follows at once.

Let now $X = R_1$. From Theorem 8 and the Radon–Nikodym theorem (see, for example, ([5]), p. 59) it follows:

Theorem 11. The space of numerical functions $W_p^l(\Omega)$, introduced by S. L. Sobolev ([2]), coincides with the space of normal additive numerical functions of a set having all generalized derivatives of order l belonging to the space $\Phi_p(R_1, \Omega) = L_p(\Omega)$ $(p > 1)$.

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Note: Figure translations are in progress. See original paper for figures.