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Abstract

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On Some Extremal Properties of Functions Multivalent in Multiply Connected Domains

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Let G be a bounded finitely connected domain of the z-plane with boundary C, consisting of simple closed analytic curves; let ζ be any prescribed point of the domain G; let $\alpha_1, \dots, \alpha_p$ be any prescribed constants, not all equal to zero; and let θ be any prescribed angle, $0 \le \theta < \pi$. Setting

$$Q_p\left(\frac{1}{z-\zeta}\right) = \sum_{k=1}^{p} \frac{\alpha_k}{(z-\zeta)^k},$$

we denote by $\varphi_{\theta,p}(z,\zeta)$ that uniquely determined function which is regular* in the domain G, except for a pole at the point $z=\zeta$, has in its neighborhood an expansion of the form

$$\varphi_{\theta,p}(z,\zeta) = Q_p\left(\frac{1}{z-\zeta}\right) + \sum_{n=1}^{\infty} a_n (z-\zeta)^n$$

and assigns to each boundary component of the domain G a segment of some straight line of inclination θ to the real axis. Let $g(z)$ be any function regular in the domain G, and let $A(g)$ be the area (finite or infinite) of the image of the domain G under the mapping $w = g(z)$ (multiply covered area is counted with multiplicity). For a function $f(z)$, regular in the domain G except for a finite number of poles, we introduce the exterior area $\overline{A}(f)$ of the function $f(z)$ in the domain G. Namely, considering a sequence of domains G_k approximating the domain G from within, we set

$$\overline{A}(f) = \lim_{k\to\infty} \frac{1}{2} \int_{B^{(k)}} R^2 \, d\Phi,$$

where $f(z) = Re^{i\Phi}$, $B^{(k)}$ is the boundary of the image W_k of the domain G_k under the mapping $w = f(z)$, and the integration is carried out in the negative direction with respect to W_k. If the function $w = f(z)$ is regular on the boundary of the domain G and assumes the value $w = \infty$ in G p times, then $\overline{A}(f)$ gives the difference between the area of the p-sheeted disk

$$|w| < \max_{z \in C} |f(z)|$$

and the area of that part of the image of the domain G which lies over it. Put

$$M_p(z,\zeta) = \frac{1}{2}\left[\varphi_{0,p}(z,\zeta) - \varphi_{\pi/2,p}(z,\zeta)\right],$$

$$N_p(z,\zeta) = \frac{1}{2}\left[\varphi_{0,p}(z,\zeta) + \varphi_{\pi/2,p}(z,\zeta)\right].$$

It is easy to see that $A(M_p) = \overline{A}(N_p)$.

Theorem 1. In the class of all functions $f(z)$, regular in the domain G except for poles at its points $z = \zeta_\nu$, $\nu = 1, \dots, s$, and having

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* In what follows, by functions regular and meromorphic in a domain we mean single-valued functions regular and meromorphic in it.

at these points the principal parts

$$Q_{p_\nu}\left(\frac{1}{z-\zeta_\nu}\right) = \sum_{k=1}^{p_\nu} \frac{\alpha_{k,\nu}}{(z-\zeta_\nu)^k},$$

the function

$$\sum_{\nu=1}^{s} N_{p_\nu}(z,\zeta_\nu)$$

and, up to an additive constant, only it gives the greatest exterior area.

This theorem generalizes the theorem known in ([1]) for univalent functions. Consider the set of all functions $M_p(z,\zeta)$ corresponding, for fixed ζ and p, to all possible systems of coefficients $\alpha_1, \dots, \alpha_p$.

Lemma. For any given system of constants $\beta_1, \dots, \beta_p$, not all simultaneously equal to zero, there exists a unique system of coefficients $\alpha_1, \dots, \alpha_p$ for which the corresponding function $M_p(z,\zeta)$ satisfies the conditions:

$$M_p^{(l)}(\zeta,\zeta) = \beta_l, \qquad l = 1, \dots, p.$$

Denoting the function $M_p^*(z,\zeta)$ indicated in this lemma by $M_p^*(z,\zeta)$, we have the theorem:

Theorem 2. Let arbitrary constants $\beta_1, \dots, \beta_p$, not all simultaneously equal to zero, be given. In the class of all functions $g(z)$, regular in the domain G and satisfying the conditions: $g^{(l)}(\zeta) = \beta_l,\ l = 1, \dots, p$, the least value of the area $A(g)$ is given by the function $M_p^*(z, \zeta)$ and, up to an additive constant, only by it.

This theorem generalizes the theorem known in ([1]) for the case $p = 1$.

Let now the function $f(z)$ be regular in the domain G except for a pole of order p at the point $z = \zeta$. We shall call the function $f(z)$ **weakly p-sheeted in area** in the domain G if for it $\overline{A}(f) \geq 0$. Denote by $\overline{\mathfrak{M}}_{p,\alpha}(G)$ the class of all functions weakly p-sheeted in area in the domain G and having fixed coefficients $\alpha_1, \dots, \alpha_p$ of the principal part

$$Q_p\left(\frac{1}{z-\zeta}\right),$$

by $\overline{\mathfrak{M}}_{p,\alpha}(G)$. It can be shown that the class $\overline{\mathfrak{M}}_{p,\alpha}(G)$ is convex. Let $\mathfrak{H}(G)$ be the class of all functions $h(z)$, regular in the domain G together with their integrals and satisfying the condition

$$\iint_G |h(z)|^2\, dx\, dy < \infty, \qquad z = x + iy,$$

and let $v_\nu(z)$, $\nu = 1, 2, \dots$, be a system of functions of the class $\mathfrak{H}(G)$, orthonormalized by the conditions

$$\iint_G v_\mu(z)\overline{v_\nu(z)}\, dx\, dy = \begin{cases} 0, & \mu \neq \nu, \\ 1, & \mu = \nu, \end{cases}$$

and complete in this class. Putting

$$(f', v_\nu) = -\frac{1}{2i}\int_C f\, \overline{v_\nu}\, d\overline{z}, \qquad \nu = 1, 2, \dots,$$

we obtain a theorem of the type of the well-known area theorem of Bieberbach:

Theorem 3. If the function $f(z)$ belongs to the class $\overline{\mathfrak{M}}_{p,\alpha}(G)$ and is regular on the boundary C of the domain G, then

$$\sum_{\nu=1}^{\infty} |(f', v_\nu)|^2 \leq \pi \sum_{k=1}^{p} \frac{\alpha_k}{(k-1)!}\, M_p^{(k)}(\zeta, \zeta).$$

The estimate is sharp.

This theorem generalizes, on the one hand, Meshkovskii' s theorem ([2]) for functions univalent in a multiply connected domain, and on the other hand the corresponding result of the author ([3]) for functions weakly p-sheeted in area in the unit disk.

Let some class $\overline{\mathfrak{M}}_{p,\alpha}(G)$ be considered. Then for any function $F(z)$ having, in a neighborhood of the point $z=\zeta$, an expansion in a power

a series with regular part of the form $\sum_{n=0}^{\infty} b_n(z-\zeta)^n$, the functional is defined by

$$J(E)=\sum_{k=1}^{p} k\alpha_k b_k.$$

Theorem 4. For any fixed θ, $0 \le \theta < \pi$, the extremal problem $\mathrm{Re}\{e^{-2i\theta}J(f)\} =$ max in the class $\mathfrak{M}_{p,\alpha}(G)$ is solved by the function $\varphi_{\theta,p}(z,\zeta)$, and, up to an additive constant, by it alone.

This theorem generalizes the corresponding result of Grunsky ([4]) for p-valent functions.

Theorem 5. If the function $f(z)$ ranges over the class $\mathfrak{M}_{p,\alpha}(G)$, then the range of values of the functional $J(f)$ is the disk $\left|w-J\left(\overline{N_p}\right)\right| \le J(M_p)$, and to each point on the boundary of this disk there corresponds a function $\varphi_{\theta,p}(z,\zeta)$ with suitable θ, and, up to an additive constant, only it.

This theorem generalizes the well-known theorem of Grötzsch ([5]) for univalent functions.

For any $\lambda \in [-1,1]$, the function

$$L_\lambda(z,\zeta)=N_p(z,\zeta)+\lambda M_p(z,\zeta)$$

belongs to the class $\mathfrak{M}_{p,\alpha}(G)$, and as λ increases from -1 to 1, $\mathrm{Re}\{J(L_\lambda)\}$, increasing, runs through the set of all values $\mathrm{Re}\{J(f)\}$ in this class.

Theorem 6. In the subclass of all functions $f(z)$ from $\mathfrak{M}_{p,\alpha}(G)$ with any fixed value $\mathrm{Re}\{J(f)\}$, the greatest exterior area in the domain G is attained by the function $N_p(z,\zeta)+\lambda M_p(z,\zeta)$ of this subclass and, up to an additive constant, only by it.

For the case where the domain G is a circular annulus, the results listed above can be made explicit. Namely, one can find that for the circular annulus $D:$ $r<|z|<1,\ r>0$, we have:

$$\varphi_{\theta,p}(z,\zeta)=Q_p\left(\frac{1}{z-\zeta}\right)+e^{2i\theta}\sum_{n=-\infty}^{\infty}{}'\frac{(\overline{\zeta}z)^n}{1-r^{2n}}\sum_{k=1}^{p}\binom{n-1}{k-1}\left(\frac{\alpha_k}{\zeta^k}\right)+$$

$$+\sum_{n=1}^{\infty}\frac{r^{2n}}{1-r^{2n}}\sum_{k=1}^{p}\frac{\alpha_k}{\zeta^k}\left[\binom{n-1}{k-1}\left(\frac{z}{\zeta}\right)^{-n}-\binom{-n-1}{k-1}\left(\frac{z}{\zeta}\right)^{n}\right]+c_0,$$

where c_0 is a suitable constant.

From this we find explicit expressions for the functions $M_p(z,\zeta)$ and $N_p(z,\zeta)$ in the case of the annulus. For $p=1$, the expressions obtained give the known functions ([1]). From Theorem 3, for the circular annulus there follows the theorem:

Theorem 7. Let $f(z)\in\mathfrak{M}_{p,\alpha}(D)$ and

$$f(z)-N_p(z,\zeta)=\sum_{n=-\infty}^{\infty}a_nz^n,\quad z\in D.$$

Then we have the sharp estimate:

$$\sum_{n=-\infty}^{\infty}n(1-r^{2n})|a_n|^2\le\sum_{n=-\infty}^{\infty}\frac{n|\zeta|^{2n}}{1-r^{2n}}\left|\sum_{k=1}^{p}\binom{n-1}{k-1}\frac{\alpha_k}{\zeta^k}\right|^2.$$

This theorem generalizes the corresponding theorem of Abe ([6]) for univalent functions in a circular annulus.

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Note: Figure translations are in progress. See original paper for figures.