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Abstract

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MATHEMATICS

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POLYIDENTICAL RELATIONS IN GROUPS

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1. Consider the free group X of countable rank with a system of free generators x_{ij}, $i, j = 1, 2, \dots, n, \dots$. Let us divide the alphabet consisting of these letters x_{ij} into subalphabets $X_1, X_2, \dots, X_i, \dots$, where X_i consists of x_{ij}, $j = 1, 2, \dots$. We agree to call words of the free group X *polywords* with respect to the indicated division of the alphabet, and we adopt the following abbreviated notation: instead of the notation $f(x_{11}, \dots, x_{1s}, \dots, x_{n1}, \dots, x_{n,s_n})$, indicating exactly which letters enter into this word, we shall write $f(x_1, \dots, x_n)$, where each x_i is no longer a letter but a certain aggregate of letters from the subalphabet X_i. Denote by VV an arbitrary but fixed set of polywords. For an arbitrary group G with some fixed system of subgroups G_α, $\alpha \in I$, generating it, $G = \{G_\alpha, \alpha \in I\}$, by its *polyverbal* VV-subgroup with respect to this system of subgroups we shall mean its normal divisor $VV(G)$ (in more detailed notation $VV\{G_\alpha, \alpha \in I\}$), generated by the values of all polywords $f(x_1, \dots, x_n) \in VV$, under the condition that all possible ordered subsystems of subgroups $G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}$, $\alpha_i \in I$, $\alpha_i \neq \alpha_j$ for $i \neq j$, are taken, and the letters $x_{i1}, \dots, x_{is_i}$, which make up x_i, are made, independently of one another, to run through the whole subgroup G_{α_i}, $i = 1, 2, \dots$. If the system of subgroups G_α is finite, then for the feasibility of the indicated construction it is permitted to supplement it with a countable set of copies of the identity group. A direct verification shows the validity of the following lemmas.

Lemma 1. For any homomorphism φ of the group $G = \{G_\alpha, \alpha \in I\}$,

$$VV\{G_\alpha, \alpha \in I\}\varphi = VV\{G_\alpha\varphi, \alpha \in I\}.$$

Lemma 2. If A_α is a subgroup of G_α, $A = \{A_\alpha, \alpha \in I\}$, then

$$VV\{A_\alpha, \alpha \in I\} \subseteq A \cap VV\{G_\alpha, \alpha \in I\}.$$

Lemma 3. If for two sets of polywords VV and VV^* the equality

$$VV\left(\prod_{i=1}^{\infty *} H_i\right) = VV^*\left(\prod_{i=1}^{\infty *} H_i\right)$$

holds, where H_i are free groups of countable ranks, then also for any group $G = \{G_\alpha, \alpha \in I\}$

$$VV\{G_\alpha, \alpha \in I\} = VV^*\{G_\alpha, \alpha \in I\}.$$

Examples. If one considers the group G as a group generated by itself, and as the polywords constituting VV takes only "monowords," then $VV(G)$ coincides with the ordinary verbal subgroup of the group G. If as the set VV one takes the single biword $[x, y] = x^{-1}y^{-1}xy$, then $VV\{G_\alpha, \alpha \in I\}$ will be the normalized mutual commutant $[G_\alpha, \alpha \in I]^G$ of the subgroups G_α.

We shall agree to say that in the group $G = \{G_\alpha, \alpha \in I\}$, with respect to the system of subgroups $G_\alpha, \alpha \in I$, there hold **polyidentity relations**

$$f(x_1, x_2, \dots, x_n) = 1, \qquad f \in VV, \tag{1}$$

if the values of all polywords from VV are equal to 1 when the x_{ij} are replaced by arbitrary elements g_{ij}, $g_{ij} \in G_{\alpha_i}$, where $\alpha_1, \dots, \alpha_n$ are pairwise distinct. In other words, this means that $VV(G) = E$. In this case we shall also say that the subgroups $G_\alpha, \alpha \in I$, are in the VV-relation. Two systems of polyidentities (1) and

$$f^*(x_1, x_2, \dots, x_m) = 1, \qquad f^* \in VV^*, \tag{2}$$

are called **equivalent** if, for every group G and every choice of subgroups generating it, $G_\alpha, \alpha \in I$, either both systems (1) and (2) hold, or both do not hold.

Lemma 4. *Two systems (1) and (2) are equivalent if and only if any one of the following conditions is satisfied:*

a) for every group $G = \{G_\alpha, \alpha \in I\}$ the equality $VV(G) = VV^*(G)$ holds;

b) for free groups of countable ranks $H_1, H_2, \dots, H_i, \dots$ the equality holds

$$VV\left(\prod_{i=1}^{\infty *} H_i\right) = VV^*\left(\prod_{i=1}^{\infty *} H_i\right).$$

We shall call a polyword $f(x_1, x_2, \dots, x_n)$ **neutral** if, in the free group X with the system of free generators x_{ij}, it becomes the empty word after an arbitrary replacement in it by units of the elements of all aggregates $x_1, x_2, \dots, x_n$, except for an arbitrary single aggregate. A system of polyidentities is **neutral** if all polywords that are its left-hand sides are neutral.

2. Let us agree on the terminology that will be used in questions concerning operations on the class of all groups. By an operation (we shall denote it by the symbol $\circ$) on the class of all groups we shall mean a rule assigning to an arbitrary system of groups $G_\alpha, \alpha \in I$ (among which there may also be isomorphic ones) a uniquely determined group

$$G = \prod_{\alpha \in I}^{\circ} G_\alpha,$$

called their product. An operation is **exact** if

$$G = \{G'_\alpha, \alpha \in I\},$$

where $G'_\alpha \simeq G_\alpha$, so that for an exact operation one may assume that $G = \{G_\alpha, \alpha \in I\}$. An exact operation is **proper** if, for each α,

$$G_\alpha \cap \{G_\beta, \beta \in I,\ \beta \neq \alpha\}^G = E$$

(A^G denotes the normal divisor generated in G by its subgroup A). An exact operation satisfies the **Mac Lane postulate** if any system of epimorphisms φ_α :

$$G_\alpha \to H_\alpha, \qquad \alpha \in I,$$

generates an epimorphism $\varphi : G \to H$, where

$$H = \prod_{\alpha \in I}^{\circ} H_\alpha,$$

and

$$\operatorname{Ker} \varphi = \{\operatorname{Ker} \varphi_\alpha,\ \alpha \in I\}^G.$$

An exact operation satisfies the **postulate of endomorphism extendability** if any system of endomorphisms $\psi_\alpha : G_\alpha \to G_\alpha, \alpha \in I$, generates an endomorphism $\psi : G \to G$. An exact operation satisfies the **weakened Mal' cev postulate** if any subgroup of the form $A_G = \{A_\alpha, \alpha \in I\}$, where $A_\alpha \leq G_\alpha$, is a factor group of

$$A = \prod_{\alpha \in I}^{\circ} A_\alpha$$

by a normal divisor lying in the normalized mutual commutant $[A_\alpha, \alpha \in I]^A$.

By the **product** of a certain system of groups $G_\alpha, \alpha \in I$, **defined by the given system of polyidentities** (1),

or, more briefly, by their VV**-product** we mean the factor group

$$G = \prod_{\alpha \in I}^{VV} G_\alpha = F/VV(F),$$

where F is their free product and the polyverbal subgroup $VV(F)$ is taken relative to the system of subgroups $G_\alpha,\ \alpha \in I$. In other words, the group G is constructed on the combined system of generators of all the groups $G_\alpha,\ \alpha \in I$, on which, in addition to the relations binding them in each group G_α, further relations are imposed that follow from the polyidentities (1).

By a VV**-operation**, defined on the class of all groups, we shall mean the operation that assigns to each system of groups their VV-product. A VV-operation, generally speaking, is not only not regular, but also not exact. We agree to call a VV-operation **neutral** if the system of identities defining it is neutral.

Theorem 1. *A VV-operation is exact if and only if it is neutral. In that case it will automatically be regular.*

Theorem 2. *An operation on the class of all groups is exact, commutative, and satisfies Mac Lane’ s postulates and the amalgamation of endomorphisms if and only if it is a neutral VV-operation.*

We note that by commutativity of an operation one means the unorderedness of the system of factors of the product.

Corollary 1. *There are no more than continuum many distinct exact commutative operations satisfying Mac Lane’ s postulates and the amalgamation of endomorphisms.*

As N. R. Brumberg observed, every exact operation satisfying Mac Lane’ s postulate is regular. Therefore we have:

Corollary 2. *Operations satisfying the requirements of Theorem 2 are regular.*

Theorem 3. *A neutral VV-operation satisfies the weakened Malʹ cev postulate.*

Theorem 4. *For an exact commutative operation satisfying Mac Lane’ s postulate, the requirements of the postulate of amalgamation of endomorphisms and of the weakened Malʹ cev postulate are equivalent.*

Corollary. *All of Moran' s verbal products* (^{1},^{2}) *are determined by certain systems of polyidentities.*

For the special case of verbal products defined by arbitrary compound commutators, this was in fact established (though in another language) by Moran (^{3}) and Struik (^{4}). The results of the present paper give, in particular, a principally positive answer to the question posed by Moran at the very end of his paper (^{3}) and concerning the verbal product defined by a word of the form x^n.

The simplest examples of operations determined by systems of polyidentities are the following. The free multiplication of groups is determined by the empty set VV, the direct multiplication by the single binary word $[x, y]$. Metabelian multiplication (^{5}) is determined by two polywords: the ternary word $[[x, y], z]$ and the binary word $[[x, y], y']$. Any n-th nilpotent multiplication (^{6}) is determined by one $(n + 2)$-ary word $[\dots[[x_0, x_1], x_2], \dots, x_{n+1}]$ and by the polywords derived from it, obtained from it by all possible replacements of certain letters x_i by letters from one and the same subalphabet, with the sole restriction that x_0 and x_1 remain belonging to different subalphabets.

3. For a fixed system of groups G_α, $\alpha \in I$, one may speak of the **"polymanifold"** $\mathfrak{M}$ of groups generated by them, determined by a given system of neutral polywords VV. To $\mathfrak{M}$ should be assigned those and only those groups each of which is generated by subgroups respectively isomorphic to the given ones, and in which, relative to the system of these subgroups, all the polyidentities determined by the set VV are fulfilled.

For every such group G one has $VV(G) = E$, and it will be a factor group of the uniquely determined "maximal" ("free") group of this polyvariety, namely

$$\prod_{\alpha\in I}^{VV} G_\alpha.$$

Such notions as the mutual nilpotence (or solvability), of a definite class, of a given system of subgroups of some group have a substantive meaning. In general, whatever the set of polyslaws VV may be, one may say that the subgroups G_α, $\alpha \in I$, of the group G are in mutual VV-relation if the subgroup generated by them belongs to the corresponding polyvariety.

4. The passage from the operation of direct multiplication of groups to VV-operations (in particular, to verbal operations) means, as we have seen, replacing the defining identity $[x, y] = 1$ by some other polyidentity or by an entire set of polyidentities. Another path is also possible: the polyidentity $[x, y] = 1$ is retained, but the field of its applicability is narrowed. It is required, say, in the case of two groups A and B, that x and y range not over the groups A and B in their entirety, but over certain specified subgroups of them. This is precisely how the semicommutative products

of Lyapin–Fridman ($^{7-9}$) and Moran' s N^*-products (^{10}) are constructed. It is clear that a synthesis of these two devices is also possible.

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Note: Figure translations are in progress. See original paper for figures.

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