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Abstract

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MATHEMATICAL PHYSICS

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ON THE SHORT-WAVE ASYMPTOTICS OF THE GREEN FUNCTION FOR THE EXTERIOR OF A BOUNDED CONVEX DOMAIN

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Let the function $u(x, y, x_0, y_0, k)$ be the solution of the following problem:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u = -\delta(x - x_0, y - y_0);$$

$$(x_0, y_0) \text{ outside } S; \qquad \left.\frac{\partial u}{\partial n}\right|_S = 0; \qquad \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right)_{r=\sqrt{x^2+y^2}\to\infty} \longrightarrow 0. \quad (1)$$

Here S is a sufficiently smooth closed convex contour, outside which the solution of problem (1) is sought; δ is the Dirac delta function. The paper studies the behavior of the function $u(x, y, x_0, y_0, k)$ as $k \to \infty$.

1°. The asymptotics of u are different in different regions of the x, y-plane. Behind the contour S a shadow region is formed, separated from the illuminated region by two rays l_1 and l_2, which are tangent to the contour S at certain points A and B and go off to infinity. The continuations of l_1 and l_2 into the illuminated region pass through the source M_0.

We shall denote the illuminated part of the contour by AmB, and the shadow part by AnB. The asymptotics of u are essentially different in the illuminated region, in the shadow region, and in the neighborhood of the limiting rays l_1 and l_2.

We shall first find the asymptotics for u when $x, y \in S$. Using the methods of work ([1]), one can construct a function $L(x, y, s_0, k)$ $(s_0 \in S)$ having the following properties:

$$(\Delta + k^2)L = 0; \qquad \left.\frac{\partial L}{\partial n}\right|_{x,y\in S} = \delta(s - s_0) + K(s, s_0, k); \qquad (2)$$

$$|K| \le C_1 \left(\exp\left(-Ck^{1/3}|s-s_0|\right) + \exp\left(-Ck^{1/7}\right)\right). \tag{3}$$

Here s and s_0 are points on S; $|s-s_0|$ is the length of the arc connecting s and s_0. The function $L(x,y,s_0,k)$ is expressed in a rather complicated way in terms of the solution of problem (1) for the exterior of a circle. In estimate (3), and in the subsequent estimates, constants C, C_1, C_2, etc., occur. Unless the contrary is specifically stated, the C_i are completely determined by the contour S and by the position of the source M_0.

Applying Green' s formula to the desired function u and to the function L, we obtain without difficulty, for

$$u(s,k) = u(x,y,x_0,y_0,k)\big|_{x,y\in S},$$

the integral equation

$$u(s_0,k) + \int_S K(s,s_0,k)u(s,k)\,ds = L(x_0,y_0,s_0,k), \tag{4}$$

whose kernel, by virtue of estimates (3), will be small.

Construct the circle of curvature for the contour S corresponding to the point s_0. Denote the solution of problem (1) for the exterior of this circle of curvature by $\Phi(x,y,s_0,k)$. Using the explicit expression for L, it is not difficult to show that, up to a term of order $\exp(-Ck^{-1/7})$, the right-hand side in the inte-

in the integral equation (4) may be replaced by $\Phi(x(s_0),y(s_0),s_0,k)$ ($x(s_0),y(s_0)$ are the Cartesian coordinates of the point $s_0 \in S$).

The integral equation (4) can be solved by the method of successive approximations:

$$u(s_0) = \Phi(x(s_0),y(s_0),s_0,k) + K\Phi + K^2\Phi + \cdots + O(\exp(-Ck^{1/7}));$$

$$K\Phi = \int K(s,s_0,k)\Phi(\ldots)\,ds; \qquad K^n\Phi = K(K^{n-1}\Phi). \tag{5}$$

Estimate (3) and formula (5) give

$$u(s_0) = \Phi(x(s_0),y(s_0),s_0,k) + R, \tag{6}$$

where

$$|R| \le C_2 \max_{s_0 \in S} |\Phi(x(s_0), y(s_0), s_0, k)| \, \frac{1}{k^{1/8}}. \tag{7}$$

Using the explicit form of Φ and of the kernel K, the estimate for the remainder term R can be considerably improved. This improved estimate has the form:

$$|R| \le \begin{cases} C_3 k^{-2-1/6}, & s \in AmB, \quad \min(|s-s_A|, |s-s_B|) \ge \alpha, \\ C_4 k^{-1/2-1/3}, & s \in AmB, \quad \min(|s-s_A|, |s-s_B|) \le \alpha, \\ C_5 k^{-1/2-1/3} (\exp(-Ck^{1/3}|s-s_A|) + \exp(-Ck^{1/3}|s-s_B|) + \\ \qquad + \exp(-Ck^{1/7})). & \end{cases} \tag{8}$$

Here α is some positive constant independent of k; C_3 depends on α. We note that the function Φ on the arc AnB has the estimate

$$|\Phi| \le C_6 k^{-1/2} (\exp(-Ck^{1/3}|s-s_A|) + \exp(-Ck^{1/3}|s-s_B|)).$$

It follows from this formula that in the region of deep shadow (i.e., for $s \in AnB$, $\min(|s_A - s|, |s_B - s|) \ge \alpha$, where α does not depend on k) the function majorizing R exceeds $|\Phi|$. Thus, in the deep shadow we obtain not an asymptotic formula for u, but only the estimate

$$|u| \le C_7 \exp(-Ck^{1/7}), \qquad s \in AnB, \qquad \min(|s-s_A|, |s-s_B|) \ge \alpha.$$

2°. In the illuminated region, for the function u it is easy to construct (formally) an asymptotic expansion. Represent u in the form

$$u = \frac{i}{4} H_0^{(1)}(kr) + g,$$

where g is the reflected part of the Green' s function; $H_0^{(1)}$ is the Hankel function. Replace the function $H_0^{(1)}(kr)$ by its asymptotic series, assuming that $kr \gg 1$. We shall also seek the function g in the form

$$g = \frac{e^{ik\tau(x,y)}}{\sqrt{k}} \sum_{l=0}^{\infty} \frac{v_l(x,y)}{k^l}. \tag{9}$$

We require that the series (9) formally satisfy the Helmholtz equation and that, on the surface S, for the series (9) and the asymptotic series for

$$\frac{i}{4} H_0^{(1)}(kr)$$

the boundary condition be fulfilled:

$$\frac{\partial}{\partial n}\left(\frac{i}{4}H_0^{(1)}+g\right)\Big|_S=0.$$

The coefficients of the series (9) will be uniquely determined by these conditions if the point x, y is in the illuminated region. Such formal constructions have been carried out by many authors (an extensive bibliography is given in [?]). It is natural to expect that the expansion (9) is an asymptotic series for g.

3°. Using the explicit expression for Φ, it is not difficult to show that in the illuminated region for $\Phi(x(s_0), y(s_0), s_0, k)$ there is an asymptotic expansion of the form (9), the first two terms of which coincide with the first two

terms of the expansion obtained by the method just described for the function

$$\frac{i}{4}H_0^{(1)}(kr)+g\Big|_S.$$

It follows from the estimates (8) that

$$u\Big|_S=e^{ikr(s)}\left(\frac{u_0(x,y)}{\sqrt{k}}+\frac{u_1(x,y)}{k\sqrt{k}}\right)+O\left(k^{-2-\frac{1}{6}}\right), \tag{10}$$

$$s\in AmB, \qquad \min(|s-s_A|,\ |s-s_B|)\geq a,$$

$$r(s)=\sqrt{(x_0-x(s))^2+(y_0-y(s))^2}.$$

Here u_0 and u_1 are obtained by means of the formal constructions of item 2°.

4°. In the case when $x, y\bar{\in}S$,

$$u(x,y,x_0,y_0,k)=\frac{i}{4}H_0^{(1)}(kr)-\int_S\frac{i}{4}\frac{\partial}{\partial n}H_0^{(1)}(kr_1)\,u(s)\,ds, \tag{11}$$

$$u(s)=u(x,y,\ldots)\Big|_{x,y\in S},$$

$$r=\sqrt{(x-x_0)^2+(y-y_0)^2}, \qquad r_1=\sqrt{(x-x(s))^2+(y-y(s))^2}.$$

Using formulas (5), (8), (10) and the explicit form of K and Φ, one can show that in the illuminated region

$$u(x, y, x_0, y_0, K) = \frac{i}{4} H_0^{(1)}(kr) + \frac{e^{ik\tau(x,y)}}{\sqrt{k}} \left(v_0 + \frac{v_1}{k} \right) + O\left(k^{-5/3}\right). \qquad (12)$$

Here v_0 and v_1 are the same as in formula (9); the estimate $O(k^{-5/3})$ is uniform for points (x, y) lying in any finite closed subdomain of the illuminated region having no common points with the limiting rays.

If the point (x, y) belongs to the shadow zone, then from formula (11) one can derive only the estimate

$$u = O\left(k^{-5/3}\right). \qquad (13)$$

Estimate (13) is uniform for points (x, y) belonging to any finite closed domain, together with its boundary, lying in the shadow zone and having no common points with the limiting rays.

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Note: Figure translations are in progress. See original paper for figures.