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Abstract

Full Text

MATHEMATICS

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ON A QUEUEING PROBLEM FOR A BATCH FLOW OF CUSTOMERS

(Presented by Academician A. N. Kolmogorov, February 23, 1962)

A batch flow of customers arrives at a service facility. Concerning the flow and the service system we shall make the following assumptions:

1. The arrival epochs of customers t_1, t_2, \dots ($0 < t_1 < t_2 < \dots$) are such that the differences $x_n = t_n - t_{n-1}$, $t_0 = 0$, form a sequence of independent random variables having one and the same distribution $A(x)$.
2. At each of the epochs t_n , with probability φ_r , r ($r = 1, 2, \dots$) customers arrive.
3. Customers that find the facility busy join the queue after all previously arrived customers; customers of one group are served one at a time in an arbitrary order.
4. The service time of a customer is a random variable y with distribution $B(x)$.
5. If customers arrive at a moment when the facility is free, then service begins after a random time z^* , distributed according to the law $C(x)$.
6. The quantities x_n , y , and z are mutually independent, and

$$a = \int_0^{\infty} x dA(x) < \infty, \quad b = \int_0^{\infty} x dB(x) < \infty, \quad c = \int_0^{\infty} x dC(x) < \infty.$$

Let W_r denote the duration of the wait for the beginning of service of the group of customers arriving at time t_r , and let N_s denote the length of the queue at the moment of departure from the service system of the s -th served customer. Further put

$$d = \sum_1^{\infty} r\varphi_r.$$

Theorem 1. *If $bd < a$, then the limiting distributions exist*

$$F(x) = \lim_{r \rightarrow \infty} P\{W_r \leq x\},$$

$$p_i = \lim_{s \rightarrow \infty} P\{N_s = i\}, \quad i = 0, 1, 2, \dots$$

The function $F(x)$ does not depend on the distribution of W_1 and (in the class of distribution functions) is the unique solution of the equation

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ \int_{-\infty}^x F(x-y) dU(y) - (1-C(x)) \int_{-\infty}^0 F(-y) dU(y), & \text{if } x \geq 0, \end{cases} \quad (1)$$

* In applied questions this corresponds to the “warm-up” time of the service facility.

where

$$U(x) = \int_0^{\infty} B_1(x+y) dA(y), \quad B_1(x) = \sum_{i=1}^{\infty} \varphi_i B^{*(i)}(x),$$

$$B^{*(1)}(x) = B(x), \quad B^{*(i)}(x) = \int_0^{\infty} B^{*(i-1)}(x-y) dB(y) \quad (i = 2, 3, \dots).$$

The proof of this theorem is a simple modification of the arguments of D. Lindley⁽⁴⁾ and P. Finch⁽⁵⁾.

Introduce the notation:

$$P(z) = \sum_0^{\infty} p_i z^i, \quad \tilde{F}(t) = \int_{0-}^{\infty} e^{itx} dF(x),$$

$$\tilde{B}(t) = \int_{0-}^{\infty} e^{itx} dB(x), \quad \tilde{C}(t) = \int_{0-}^{\infty} e^{itx} dC(x),$$

$$\varphi(z) = \sum_1^{\infty} \varphi_i z^i, \quad \psi_i = \sum_{s=i}^{\infty} \frac{\varphi_s}{s}, \quad \psi(z) = \sum_1^{\infty} \psi_i z^{i-1}.$$

Theorem 2. If $A(x) = 1 - e^{-\lambda x}$, $\lambda b d < 1$, then

$$P(z) = \frac{1 - \lambda bd}{(1 + \lambda c)d} \frac{[\tilde{C}(i\lambda - i\lambda\varphi(z))\varphi(z) - 1]\tilde{B}(i\lambda - i\lambda\varphi(z))}{z - \tilde{B}(i\lambda - i\lambda\varphi(z))}; \quad (2)$$

$$\tilde{F}(t) = \frac{1 - \lambda bd}{1 + \lambda c} \frac{(\lambda + it)\tilde{C}(t) - \lambda}{\lambda + it - \lambda\varphi(\tilde{B}(t))}. \quad (3)$$

Proof. The quantities N_s and N_{s+1} are related by

$$N_{s+1} = \begin{cases} N_s + \xi_{s+1} - 1, & \text{if } N_s > 0, \\ \zeta + \xi + \xi_{s+1} - 1, & \text{if } N_s = 0. \end{cases} \quad (4)$$

Here ξ_{s+1} is the number of customers arriving during the service time of the $(s + 1)$ -st customer; ζ is the size of the group arriving at the moment when the server was free; ξ is the number of customers arriving during the waiting time before service of this group began. The generating function of the compound Poisson flow has the form $Mz^{\nu(t)} = \exp[\lambda t(\varphi(z) - 1)]$, where $\nu(t)$ is the number of customers of the flow that arrived during time t . Hence

$$Mz^{\xi_{s+1}} = \tilde{B}(i\lambda - i\lambda\varphi(z)), \quad Mz^\xi = \tilde{C}(i\lambda - i\lambda\varphi(z)). \quad (5)$$

From (4) it follows that

$$zMz^{N_{s+1}} = \mathbf{P}\{N_s > 0\}M_{N_s > 0}z^{N_s + \xi_{s+1}} + \mathbf{P}\{N_s = 0\}M_{N_s = 0}z^{\zeta + \xi + \xi_{s+1}}. \quad (6)$$

But

$$\mathbf{P}\{N_s > 0\}M_{N_s > 0}z^{N_s} = Mz^{N_s} - \mathbf{P}\{N_s = 0\}. \quad (7)$$

Substituting (5) and (7) into (6) and passing to the limit as $s \rightarrow \infty$, we obtain

$$P(z) = \frac{[\tilde{C}(i\lambda - i\lambda\varphi(z))\varphi(z) - 1]\tilde{B}(i\lambda - i\lambda\varphi(z))p_0}{z - \tilde{B}(i\lambda - i\lambda\varphi(z))}. \quad (8)$$

The unknown p_0 is found from the normalization condition $\sum_0^\infty p_i = 1$. Applying L' Hôpital' s rule to (8), recalling here that

$$\tilde{A}'(0) = ia, \quad \tilde{B}'(0) = ib, \quad \tilde{C}'(0) = ic, \quad \varphi'(1) = d,$$

we find

$$p_0 = \frac{1 - \lambda bd}{(1 + \lambda c)d}.$$

To solve equation (1), consider the function

$$g(x) = \int_{-\infty}^x F(x - y) dU(y) \quad (-\infty < x < \infty), \quad (9)$$

where

$$U(x) = \lambda \int_0^{\infty} B_1(x + y) e^{-\lambda y} dy. \quad (10)$$

According to (1) and (10),

$$g(x) = \begin{cases} \alpha e^{\lambda x}, & \text{if } x < 0, \\ \alpha(1 - C(x)) + F(x), & \text{if } x \geq 0, \end{cases} \quad (11)$$

where

$$\alpha = \int_{-\infty}^0 F(-y) dU(y).$$

From (9) and (10),

$$\int_{-\infty}^{\infty} e^{itx} dg(x) = \frac{\lambda}{\lambda + it} \tilde{F}(t) \varphi(\tilde{B}(t)),$$

and from (11),

$$\int_{-\infty}^{\infty} e^{itx} dg(x) = \frac{\lambda \alpha}{\lambda + it} - \alpha \tilde{C}(t) + \tilde{F}(t).$$

Comparing the last two equalities gives

$$\tilde{F}(t) = \frac{(\lambda + it)\tilde{C}(t) - \lambda}{\lambda + it - \lambda\varphi(\tilde{B}(t))} \alpha.$$

Since $\tilde{F}(0) = 1$, we have

$$\alpha = \frac{1 - \lambda bd}{1 + \lambda c}.$$

Let now W be the waiting time until the beginning of service of an arbitrarily chosen customer, and let $W(x) = P\{W \leq x\}$. If ν is the ordinal number of the service of the customer under consideration within the group to which it belongs, then

$$W = \gamma + \sum_{r=1}^{\nu-1} \gamma_r,$$

where

$$P\{\gamma < x\} = F(x), \quad P\{\gamma_r < x\} = B(x).$$

By the formula of total probability,

$$P\left\{\sum_{r=1}^{\nu-1} \gamma_r < x\right\} = \sum_{i=1}^{\infty} \psi_i B^{*(i-1)}(x),$$

Hence

$$\widetilde{W}(t) = \widetilde{F}(t) \psi(\widetilde{B}(t)).$$

Consequently,

$$\widetilde{W}(t) = \frac{1 - \lambda b d}{1 + \lambda c} \frac{[(\lambda + it)\widetilde{C}(t) - \lambda]\psi(\widetilde{B}(t))}{\lambda + it - \lambda\varphi(\widetilde{B}(t))}. \quad (12)$$

The formulas (2) and (12) obtained by us contain, as special cases, the corresponding results of papers (1-7). If $P\{z = 0\} = 1$ and $\varphi_i = \delta_{ri}$, $i = 1, 2, \dots$, then

$$P(z) = \frac{(r^{-1} - \lambda b)(z^r - 1)\widetilde{B}(i\lambda - i\lambda z^r)}{z - \widetilde{B}(i\lambda - i\lambda z^r)},$$

$$\widetilde{W}(t) = \frac{i(r^{-1} - \lambda b)t \sum_0^{r-1} (\widetilde{B}(t))^k}{\lambda + it - \lambda(\widetilde{B}(t))^r}.$$

Similar formulas were obtained by F. Foster (8) in the case when $B(x) = 1 - e^{-x/b}$, $A(x)$ is an arbitrary distribution function with finite mathematical expectation, and $\varphi_i = \delta_{ri}$, $i = 1, 2, \dots$

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