ON THE EXISTENCE THEOREM FOR THE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED TYPE

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Abstract

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MATHEMATICS

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ON THE EXISTENCE THEOREM FOR THE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED TYPE

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In the present note an existence theorem is proved for a boundary-value problem for the M. A. Lavrent' ev—A. V. Bitsadze equation

$$\frac{\partial^2 z}{\partial x^2} + \operatorname{sgn} y \, \frac{\partial^2 z}{\partial y^2} = 0. \tag{1}$$

§ 1. Consider a domain Δ, bounded in the half-plane $y > 0$ by the half-lines OA, $x = 0$, and DE, $x = 1$, and by a certain smooth curve DC joining the points $C(m, 0)$ and $D(1, y)$, and by two characteristics OB, $y = -x$, and CB, $y = x - m$ $(0 < m < 1)$, in the half-plane $y < 0$.

Statement of the F. I. Frankl problem for the M. A. Lavrent' ev—A. V. Bitsadze equation. Find a solution of equation (1) in the domain Δ $(y \neq 0)$, continuous up to the boundary, bounded at infinity, having continuous partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ in Δ; moreover, near the points O and C they may tend to infinity of order less than one, and satisfying on the boundary of the domain the conditions:

$$\begin{aligned} &\text{on the characteristic } OB \qquad z = 0; \\ &\text{on the half-lines } OA \text{ and } DE \qquad z = 0; \\ &\text{on } CD \qquad a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = 0. \end{aligned} \tag{2}$$

§ 2. **Theorem.** *In the domain Δ there exists a solution of equation (1) satisfying the conditions listed above.*

Proof. We construct the Green function of the Frankl problem for the Lavrent' ev—Bitsadze equation. In the lower half-plane the solution, as is known, has the form

$$u(x, y) = f(x + y) + f_1(x - y), \tag{3}$$

where f and f_1 are arbitrary twice differentiable functions.

On the characteristic $y=-x$, $u=0$; therefore $f(0)+f_1(2x)=0$, whence

$$u(x,y)=f(x+y)-f(0). \tag{4}$$

Differentiating (4) with respect to x and to y and letting y tend to zero, we obtain

$$u'_x(x,0)=\tau'(x), \qquad u'_y(x,0)=\nu(x),$$

whence

$$\tau'(x)=\nu(x). \tag{5}$$

The obtained condition (5) is equivalent to the condition prescribed on the characteristic ($z=0$).

By direct verification it is easy to see that the function

$$u(x,y)=\ln\sqrt{(x-x_0)^2+(y-y_0)^2}-\operatorname{arc\,tg}\frac{x-x_0}{y+y_0} \tag{6}$$

is a solution of equation (1) satisfying condition (5).

To construct a solution satisfying the zero conditions on OA and DE, we use the method of reflection. Taking two terms of the series constructed in this way, we obtain the function

$$\begin{aligned} g^* = {} & \ln\sqrt{(x-x_0)^2+(y-y_0)^2}-\operatorname{arc\,tg}\frac{x-x_0}{y+y_0}-\ln\sqrt{(x-x_0)^2+(y-y_0)^2} \\ & +\operatorname{arc\,tg}\frac{x+x_0}{y+y_0}+\ln\sqrt{(x-2-x_0)^2+(y-y_0)^2}-\operatorname{arc\,tg}\frac{x-2-x_0}{y+y_0} \\ & -\ln\sqrt{(x-2+x_0)^2+(y-y_0)^2}+\operatorname{arc\,tg}\frac{x-2+x_0}{y+y_0}, \end{aligned} \tag{7}$$

which on the half-lines OA, $x=0$, and DE, $x=1$, $(y>0)$, takes, respectively, the values:

$$\begin{aligned} g^*(0,y,x_0,y_0) = {} & \ln\sqrt{(2+x_0)^2+(y-y_0)^2}-\ln\sqrt{(2-x_0)^2+(y-y_0)^2} \\ & +2\operatorname{arc\,tg}\frac{x_0}{y+y_0}+\operatorname{arc\,tg}\frac{2+x_0}{y+y_0}-\operatorname{arc\,tg}\frac{2-x_0}{y+y_0}, \end{aligned}$$

$$g^*(1, y, x_0, y_0) = 2\left(\operatorname{arc\,tg}\frac{1+x_0}{y+y_0} - \operatorname{arc\,tg}\frac{1-x_0}{y+y_0}\right).$$

The functions obtained by us, $g^*(0, y, x_0, y_0)$ and $g^*(1, y, x_0, y_0)$, are analytic for all y, y_0 $(y, y_0 > 0)$ and $0 \le x \le 1$.

Let $\varphi(x, y, x_0, y_0)$ be the solution of Tricomi's problem for the domain considered in the problem, vanishing on the characteristic $y = -x$ and taking the values:

$$\begin{aligned}&\text{on } OA \quad \varphi(0, y, x_0, y_0) = g^*(0, y, x_0, y_0),\\ &\text{on } DE \quad \varphi(1, y, x_0, y_0) = g^*(1, y, x_0, y_0).\end{aligned}$$

Then the function

$$g(x, y, x_0, y_0) = g^*(x, y, x_0, y_0) - \varphi(x, y, x_0, y_0) \tag{8}$$

will be the Green' s function of the Frankl problem for the Lavrent' ev-Bitsadze equation. The existence of the function $\varphi(x, y, x_0, y_0)$ follows from work (^{1}).

§ 3. We shall seek the solution of the posed problem in the form

$$z = \int_L \mu(s) g(x, y, x_0, y_0)\, ds. \tag{9}$$

Therefore

$$\frac{\partial z}{\partial x} = \int_L \mu(s)\left(\frac{\partial g^*}{\partial x} - \frac{\partial \varphi}{\partial x}\right) ds, \qquad \frac{\partial z}{\partial y} = \int_L \mu(s)\left(\frac{\partial g^*}{\partial y} - \frac{\partial \varphi}{\partial y}\right) ds,$$

where L denotes the arc CD. More explicitly:

$$\frac{\partial z}{\partial x} = \int_L \mu(s)\left\{\frac{x - x_0}{(x-x_0)^2 + (y-y_0)^2} - \frac{x + x_0 - 2}{(x+x_0-2)^2 + (y-y_0)^2} + P(x, y, x_0, y_0)\right\},$$

$$\frac{\partial z}{\partial y} = \int_L \mu(s)\left\{\frac{y - y_0}{(x-x_0)^2 + (y-y_0)^2} - \frac{y - y_0}{(x+x_0-2)^2 + (y-y_0)^2} + Q(x, y, x_0, y_0)\right\}. \tag{10}$$

Here $P(x, y, x_0, y_0)$ and $Q(x, y, x_0, y_0)$ denote sums of regular terms.

Let the point (x_0, y_0) tend to the boundary L. In doing so we use the limiting-transition formulas

$$\frac{\partial z_i}{\partial x} = \frac{\partial z_0}{\partial x} + \pi\mu_0 \cos(nx), \qquad \frac{\partial z_i}{\partial y} = \frac{\partial z_0}{\partial y} - \pi\mu_0 \cos(ny), \tag{11}$$

where the subscript zero indicates that the value is taken on the boundary L.

Introduce new notation

$$x_0 = x(s); \qquad y_0 = y(s); \qquad x = x(\sigma); \qquad y = y(\sigma)$$

and, applying formulas (11) to (10), substitute the obtained values of the partial derivatives into the last of the boundary conditions (2). We arrive at the singular integral equation:

$$\begin{aligned}
&\pi\{a(\sigma)\cos(nx) + b(\sigma)\cos(ny)\}\mu(\sigma) + a(\sigma)\int_L \mu(s)\frac{[x(\sigma) - x(s)]\,ds}{[x(\sigma) - x(s)]^2 + [y(\sigma) - y(s)]^2}\\
&\quad + b(\sigma)\int_L \mu(s)\frac{[y(\sigma) - y(s)]\,ds}{[x(\sigma) - x(s)]^2 + [y(\sigma) - y(s)]^2}\\
&\quad - a(\sigma)\int_L \mu(\sigma)\frac{[x(\sigma) + x(s) - 2]\,ds}{[x(\sigma) + x(s) - 2]^2 + [y(\sigma) - y(s)]^2}\\
&\quad - b(\sigma)\int_L \mu(s)\frac{[y(\sigma) - y(s)]\,ds}{[x(\sigma)x(s) - 2]^2 + [y(\sigma) - y(s)]^2} + a(\sigma)\int_L \mu(s)P(\sigma, s)\,ds\\
&\quad + b(\sigma)\int_L \mu(s)Q(\sigma, s)\,ds = 0.
\end{aligned} \tag{12}$$

Transform the singular integrals in the left-hand side of equation (12), separating out the principal part:

$$\begin{aligned}
\int_L \mu(s)\frac{[x(\sigma) - x(s)]\,ds}{[x(\sigma) - x(s)]^2 + [y(\sigma) - y(s)]^2} &= \int_L \mu(s)\frac{K_1(\sigma, s)}{\sigma - s}\,ds\\
&= \int_L \mu(s)\frac{K_1(\sigma, \sigma)}{\sigma - s}\,ds + \int_L \mu(s)\frac{K_1(\sigma, s) - K_1(\sigma, \sigma)}{\sigma - s}\,ds\\
&= -x'(\sigma)\int_L \frac{\mu(s)}{s - \sigma}\,ds - \int_L \mu(s)\frac{K_1(\sigma, s) - x'(\sigma)}{s - \sigma}\,ds,
\end{aligned}$$

where

$$K_1(\sigma, s) = \frac{(\sigma - s)[x(\sigma) - x(s)]}{[x(\sigma) - x(s)]^2 + [y(\sigma) - y(s)]^2},$$

$$K_1(\sigma,\sigma)=\lim_{s\to\sigma}K_1(\sigma,s)=\lim_{s\to\sigma}\frac{\dfrac{x(\sigma)-x(s)}{\sigma-s}}{\left[\dfrac{x(\sigma)-x(s)}{\sigma-s}\right]^2+\left[\dfrac{y(\sigma)-y(s)}{\sigma-s}\right]^2}=x'(\sigma).$$

Similarly we transform the other four integrals entering into the left-hand side of (12). Substituting these values into (12) and taking into account that

$$a(\sigma)\cos(nx)+b(\sigma)\cos(ny)=a(\sigma)y'(\sigma)+b(\sigma)x'(\sigma),$$

we obtain

$$\begin{aligned}
&\pi\{a(\sigma)y'(\sigma)+b(\sigma)x'(\sigma)\}\mu(\sigma)-\{a(\sigma)x'(\sigma)+b(\sigma)y'(\sigma)\}\int_L\frac{\mu(s)}{s-\sigma}\,ds\\
&\quad-\{a(\sigma)x'(l)+b(\sigma)y'(l)\}\int_L\frac{\mu(s)\,ds}{\sigma-s-2l}-a(\sigma)\int_L\frac{K_1(\sigma,s)-x'(\sigma)}{s-\sigma}\mu(s)\,ds\\
&\quad-b(\sigma)\int_L\frac{K_2(\sigma,s)-y'(\sigma)}{s-\sigma}\mu(s)\,ds-a(\sigma)\int_L\frac{K_3(\sigma,s)-x'(l)}{\sigma+s-2l}\mu(s)\,ds\\
&\quad-b(\sigma)\int_L\frac{K_4(\sigma,s)-y'(l)}{\sigma+s-2l}\mu(s)\,ds+a(\sigma)\int_L P(\sigma,s)\mu(s)\,ds\\
&\quad+b(\sigma)\int_L Q(\sigma,s)\mu(s)\,ds=0.
\end{aligned}$$

Introduce the notation: $A(\sigma)=a(\sigma)y'(\sigma)+b(\sigma)x'(\sigma)$, $B(\sigma)=a(\sigma)x'(\sigma)+b(\sigma)y'(\sigma)$, $B(\sigma,l)=a(\sigma)x'(l)+b(\sigma)y'(l)$. Then the equation takes the form

$$\begin{aligned}
&A(\sigma)\mu(\sigma)-\frac{B(\sigma)}{\pi}\int_L\left\{\frac{1}{s-\sigma}+\frac{1}{\sigma+s-2l}\right\}\mu(s)\,ds-\frac{B(\sigma,l)-B(\sigma)}{\pi}\int_L\frac{\mu(s)\,ds}{\sigma+s-2l}\\
&\quad-\frac{a(\sigma)}{\pi}\int_L\frac{K_1(\sigma,s)-x'(\sigma)}{s-\sigma}\,\mu(s)\,ds-\frac{b(\sigma)}{\pi}\int_L\frac{K_2(\sigma,s)-y'(\sigma)}{s-\sigma}\,\mu(s)\,ds\\
&\quad-\frac{a(\sigma)}{\pi}\int_L\frac{K_3(\sigma,s)-x'(l)}{\sigma+s-2l}\,\mu(s)\,ds-\frac{b(\sigma)}{\pi}\int_L\frac{K_4(\sigma,s)-y'(l)}{\sigma+s-2l}\,\mu(s)\,ds\\
&\quad+\frac{a(\sigma)}{\pi}\int_L P(\sigma,s)\mu(s)\,ds+\frac{b(\sigma)}{\pi}\int_L Q(\sigma,s)\mu(s)\,ds=0.
\end{aligned}\tag{13}$$

§ 4. Let us compute the index of the singular equation obtained, using the known formulas (^{2}). As the investigation shows, if the solution of the problem is sought in the class of functions that tend to infinity at both ends of the arc L with order less than one, then the index x will be equal to one. Therefore, proceeding from the general theory (²), the singular integral equation (13) obtained by us can be regularized and has no more than one independent solution. Thus, the existence theorem is proved.

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CITED LITERATURE

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2. N. I. Muskhelishvili, *Singular Integral Equations*, 1946.

Note: Figure translations are in progress. See original paper for figures.

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