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Abstract

Full Text

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V. PONOMAREV

ON STRONGLY PARACOMPACT SPACES*

(Presented by Academician P. S. Aleksandrov, 23 XI 1961)

Let X be a normal space, and let $\overline{X}$ be some extension of it. We shall say that X is **paracompactly situated** in $\overline{X}$ if, whatever the neighborhood VX of the set X in the space $\overline{X}$, there exists a smaller neighborhood $V'X \subseteq VX$ which is a paracompactum.

Main theorem. *In order that a normal space X be strongly paracompact, it is necessary and sufficient that it be paracompactly situated in its maximal bicompact extension βX (the Stone-Čech extension).*

Preliminary remarks. 1°. If $\gamma = \{H_\alpha\}$ is some covering of the given space X, then by $\overline{\gamma} = \{[H_\alpha]\}$ we shall always denote the covering (of the same space X) whose elements are the closures $[H_\alpha]$ of the elements of the covering γ.

2°. Let H be an arbitrary open set of the space X; by OH we denote the largest open set in βX that gives, in intersection with X, the set H. As is known, sets of the form OH form a base of the space βX.

Hence (and from the regularity of the space βX) it follows at once:

Lemma 1. *If VX is an arbitrary neighborhood of the set X in the space βX, then there exists a covering $\gamma = \{\Gamma\}$ such that:*

1°. $[\Gamma]_{\beta X} \subseteq VX$ for every $\Gamma \in \gamma$.

2°. $\Gamma = O(X \cap \Gamma)$.

The theorem formulated above is contained in the following two propositions:

I. *If a completely regular space X is paracompactly situated in some of its bicompact extensions bX, then X is strongly paracompact.*

II. *If X is a strongly paracompact space, then every neighborhood VX of the set X in the space βX contains a smaller neighborhood $V'X \subseteq VX$ which is a strongly paracompact space**.*

Proof of Proposition I. Let $\omega = \{H\}$ be an arbitrary covering of the space X. Put $\Omega = \{OH\}$,

$$VX = \bigcup_{H \in \omega} OH.$$

By hypothesis, there exists a paracompact neighborhood $V'X \subseteq VX$. For each point $x \in V'X$ we take such a neighborhood Ox that its closure $[Ox]_{\beta X}$ is contained in $V'X$ and in some $OH \in \Omega$. Into the covering formed by these neighborhoods we inscribe a locally finite covering Ω_1, and we take a covering $\Omega' = \{U_\alpha\}$ of the same space $V'X$ such that $\overline{\Omega'}$ is

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* As is known, a space X is called **strongly paracompact** (or star-paracompact) if into every one of its (open) coverings ω one can inscribe a star-finite covering ω'. Yu. M. Smirnov proved ([2]) that in this definition star-finiteness can be replaced by star-countability.

** It may be noted (this will be proved later) that every paracompactum which is an open set in a bicompactum will be strongly paracompact.

combinatorially inscribed in Ω_1. Then $\overline{\Omega}'$ is a locally finite system of bicompacts (inscribed in Ω and covering $V'X$); this system is necessarily star-finite. Its elements, when intersected with X, give a star-finite covering of the space X, inscribed in ω, as was required to prove.

Proof of Proposition II.
The following is known.

Lemma 2. *Every regular space R that is the body of a star-finite system of bicompacts $\sigma = \{\Phi_\alpha\}$ (strongly paracompact spaces) is necessarily strongly paracompact.*

This lemma may be proved, for example, as follows. Following P. S. Aleksandrov (see, for example, (1), Ch. 5, Sec. 10, or (2), Lemma 2), call a **chain of elements** of the given system of sets σ any finite sequence of the form $\Phi_{\alpha_1}, \dots, \Phi_{\alpha_s}$, where $\Phi_{\alpha_i} \cap \Phi_{\alpha_{i+1}} \neq \Lambda$. A system of sets is called **connected** if any two of its elements can be joined by a chain. Every system of sets decomposes into "components," i.e. into maximal connected subsystems. If the system is star-finite, then its components are finite or countable subsystems. The system σ is a star-finite system of bicompacts; the components σ_ν are countable subsystems whose bodies are disjoint open-and-closed sets $R_\nu = \tilde{\sigma}_\nu$, the body R of the whole system σ. Let $\gamma = \{H\}$ be an arbitrary open covering of the space R; without loss of generality it may be assumed to be inscribed in the disjoint covering $\Xi = \{R_\nu\}$ of the space R. Then

$$\gamma = \bigcup_\nu \gamma_\nu,$$

where γ_ν consists of all $H \in \gamma$ that intersect R_ν (and, consequently, do not intersect any $R_{\nu'}$, $\nu' \neq \nu$). We have

$$R_\nu = \tilde{\sigma}_\nu = \bigcup_{i=1}^{\infty} \Phi_i^\nu,$$

where the Φ_i^ν are all elements of the system σ forming the subsystem σ_ν.

For each Φ_i^ν choose a finite subsystem γ_i^ν of the system γ_ν covering the bicompact Φ_i^ν; we obtain a countable covering

$$\gamma'_\nu = \bigcup_i \gamma_i^\nu$$

of the set R_ν.

Since two elements belonging respectively to the systems γ'_ν and $\gamma'_{\nu'}$, with $\nu' \neq \nu$, do not intersect, it follows that

$$\gamma' = \bigcup_\nu \gamma'_\nu$$

is a star-countable subsystem of the system γ, which is a covering of the whole space R. Thus every open covering of the space R contains a star-countable one, which, by the already mentioned theorem of Yu. M. Smirnov, means that R is strongly paracompact.

We turn to the proof of Proposition II. Let X be strongly paracompact, and let VX be an arbitrary neighborhood of the set X in the space βX. In view of Lemma 2 it is enough to construct a neighborhood $V_1X \subseteq VX$ that is the body of a star-finite system of bicompacts. According to Lemma 1, take a covering $\gamma = \{\Gamma_\nu\}$ of the set VX whose elements satisfy the conditions $\Gamma_\nu = O(X \cap \Gamma_\nu)$, $[\Gamma_\nu]_{\beta X} \subseteq VX$. Take the covering $X\gamma = \{H_\nu\}$ of the space X, consisting of all elements $H_\nu = \Gamma_\nu \cap X$. As a consequence of the strong paracompactness of the space X, there exists a covering $\omega = \{U_\alpha\}$ of the space X such that $\omega = \{[U_\alpha]_X\}$ is star-finite and inscribed in $X\gamma$. Take a covering $\omega' = \{U'_\alpha\}$ of the space X such that $\omega' = \{[U'_\alpha]_X\}$ is combinatorially inscribed in ω, so that for every α we have

$$U'_\alpha \subseteq [U'_\alpha]_X \subseteq U_\alpha \subseteq [U_\alpha]_X \subseteq H_\nu = X \cap \Gamma_\nu, \tag{1}$$

where $\nu = \nu(\alpha)$.

Obviously, $\overline{\omega}'$ is star-finite. By virtue of inclusion (1) and of the basic properties of the space βX, we have

$$[OU'_\alpha]_{\beta X} \subseteq OU_\alpha \subseteq [OU_\alpha]_{\beta X} \subseteq OH_\nu = \Gamma_\nu. \tag{2}$$

Put

$$V_1X = \bigcup_\alpha OU_\alpha;$$

from the last inclusion (2) it follows that $V_1X \subseteq VX$. Since for any two open subsets H_1 and H_2 of X one always has $OH_1 \cap OH_2 = O(H_1 \cap H_2)$, i.e., the operator O preserves the intersection scheme, the systems $\Omega = \{OU_\alpha\}$, $\Omega' = \{OU'_\alpha\}$ are star-finite. By virtue of (2), the system of bicompacts $\overline{\Omega}' = \{[OU'_\alpha]_{\beta X}\}$ is also star-finite. Since X is everywhere dense in βX, we have

$$V_1X = \left[\bigcup_\alpha U'_\alpha\right]_{V_1X} \subseteq \left[\bigcup_\alpha OU'_\alpha\right]_{V_1X}.$$

The system Ω is star-finite, and hence a fortiori locally finite in its body V_1X; since the system $\overline{\Omega}'$ is combinatorially inscribed in V_1X, it too is locally finite in V_1X; therefore

$$\left[\bigcup_\alpha OU'_\alpha\right]_{V_1X} = \bigcup_\alpha [OU'_\alpha]_{V_1X}. \tag{3}$$

But (in view of (2) and the definition of the set V_1X) we have $[OU'_\alpha]_{V_1X} = [OU]_{\beta X}$, so that (3) is rewritten in the form

$$V_1X = \bigcup_\alpha [OU'_\alpha]_{\beta X},$$

which proves everything.

Remark. The following question remains open: can the maximal bicompact extension βX in the formulation of Theorem 1 be replaced by an arbitrary bicompact extension bX of the space X?

Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

[1] P. S. Aleksandrov, P. S. Uryson, *Trudy Mat. Inst. im. V. A. Steklova AN SSSR*, **31** (1950). [2] Yu. M. Smirnov, *Izv. AN SSSR, ser. matem.*, **20**, 253 (1956).

Note: Figure translations are in progress. See original paper for figures.