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Abstract

Full Text

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ON THE QUESTION OF THE INDEX OF ELIPTIC OPERATORS

(Presented by Academician I. G. Petrovskii, 14 X 1961)

Let, in a bounded domain G of the Euclidean n -dimensional space R_x^n , for a system elliptic in the sense of I. G. Petrovskii,

$$\sum_{k=1}^N L_{ik} \left(x, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) u_k(x) = f_i(x) \quad (i = 1, \dots, N) \quad (1)$$

with complex coefficients, an elliptic boundary-value problem be posed. In this note the following question is considered: when does the index of this problem not depend on the values of the coefficients of the system (1) *inside* the domain G ? It is shown, in particular, that each of the following two conditions is sufficient: $n > N$; n is odd.

For $n = 2$ the question of the index was studied in detail by A. I. Volpert (see, for example, ⁽¹⁾). Assuming that the domain G is simply connected and that the order of the boundary condition is lower than the order of the system (1), he obtained explicit formulas for the index, expressing it in terms of certain homotopy invariants. From these formulas it is clear that, under the indicated assumptions, the index, generally speaking, depends on the coefficients of the system (1) inside the domain if $N > 1$, and does not depend on them if $N = 1$. This is why the question formulated above arose. I. M. Gel' fand posed the general problem of homotopy classification of elliptic boundary-value problems ⁽²⁾. The results of the present note give some progress in the solution of this problem. Other results were recently obtained by A. S. Dynin. In particular, one of his theorems ⁽¹²⁾, together with a theorem of B. V. Boyarskii ⁽³⁾, gives the sufficiency of our condition $n > N$ for $N = 1$.

Let us make our assumptions precise. We shall regard the boundary Γ of the domain G as infinitely differentiable*. Consider in the domain G two systems of partial differential equations:

$$L \left(x, \frac{\partial}{\partial x} \right) u(x) = f(x) \quad \text{and} \quad M \left(x, \frac{\partial}{\partial x} \right) u(x) = g(x). \quad (2)$$

Here u , f , and g are N -dimensional vectors, while L and M are square matrices of order N , whose elements L_{ij} and M_{ij} are differential operators of order $\leq m$.

Let the coefficients of these operators belong to $C^\infty(\overline{G})$ and coincide pairwise for $x \in \Gamma$. We shall assume the systems (2) elliptic in the sense of I. G. Petrovskii:

$$\det P(x, \xi) \neq 0 \quad \text{and} \quad \det Q(x, \xi) \neq 0 \quad (x \in \overline{G}, \xi \neq 0),$$

where P and Q are the characteristic matrices of these systems. Let on Γ the boundary condition

$$\lim_{x \rightarrow y} B \left(y, \frac{\partial}{\partial x} \right) u(x) = 0 \quad (y \in \Gamma), \quad (3)$$

be posed, where B is a certain rectangular matrix whose number of columns is equal to N . Consider L and M as operators from the space $W_2^l(G)$ (S. L. So-

* Here and below, the requirements of infinite smoothness in x may be replaced by requirements of finite smoothness

to the space $W_2^k(G)$, where $k = l - m \geq 0$, with domain singled out by conditions (3). These operators will be bounded. We shall assume that these operators are, in the terminology of I. C. Gohberg and M. G. Krein ⁽⁷⁾, Φ -operators. This means that each of the equations $Lu = 0$ and $Mu = 0$ has a finite number of linearly independent solutions $\alpha(L)$ and $\alpha(M)$, that the ranges R_L and R_M of the operators L and M are closed in $W_2^k(G)$, and that the quotient spaces $W_2^k(G)/R_L$ and $W_2^k(G)/R_M$ have finite dimensions $\beta(L)$ and $\beta(M)$. (Apparently, this assumption will be satisfied if the boundary Γ and the operator B are subjected to the following conditions: Γ is an $(n - 1)$ -dimensional manifold; B has infinitely differentiable coefficients on Γ ; the highest order of differentiation in B does not exceed $l - 1$; at each point of the boundary B is related to L and M by the condition of Ya. B. Lopatinskii (cf. ^(2,4-6)). In particular, the indices $\varkappa(L)$ and $\varkappa(M)$ of the operators L and M will be finite ($\varkappa = \alpha - \beta$). In the terminology of F. Browder ⁽⁶⁾, problems (2)–(3) will be elliptic boundary-value problems.

Our aim is to clarify the conditions under which $\varkappa(L) = \varkappa(M)$. Since an operator sufficiently close in norm to a given Φ -operator is a Φ -operator with the same index ⁽⁷⁾, one may, without loss of generality, assume that the coefficients of the operators L and M coincide pairwise near Γ . Put (E denotes the identity matrix)

$$\sigma(x, \xi) = \begin{cases} Q(x, \xi)P^{-1}(x, \xi), & \text{for } x \in G, \xi \neq 0, \\ E, & \text{for } x \notin \overline{G}, \xi \neq 0. \end{cases} \quad (4)$$

We shall now use the theory of multidimensional singular operators constructed by S. G. Mikhlin ⁽⁸⁾ and A. Calderón and A. Zygmund ⁽⁹⁾. Define in R_x^n the singular integral operator

$$Au = a(x)u(x) + \int K(x, x - y)u(y) dy, \quad (5)$$

whose symbol is the matrix $\sigma(x, \xi)$. Here $a(x)$ and $K(x, y)$ are square matrices of order N . The matrix $a(x)$ consists of infinitely differentiable functions and coincides with E for $x \notin G_1$, where G_1 is a domain lying inside G . The matrix $K(x, y)$ consists of functions, infinitely differentiable for $y \neq 0$, homogeneous in y of degree $-n$, whose means over the sphere $|y| = 1$ are equal to 0. For $x \notin G_1$ we have $K(x, y) = 0$. The operator A is a bounded Φ -operator in $L_2(\mathbb{R}^n)$. Denote by A_0 the restriction of the operator A to functions from $W_2^k(G)$, extended by zero outside G .

Lemma 1. The operator A_0 is a bounded Φ -operator in $W_2^k(G)$ for any $k \geq 0$. Its index $\varkappa(A_0)$ does not depend on k and is equal to $\varkappa(A)$.

This follows from (8,9).

Lemma 2. The operator $A_0L - M$ is a completely continuous operator from $W_2^l(G)$ to $W_2^k(G)$.

This is proved by means of a partition of unity and the Fourier transform.

From Lemmas 1 and 2 it follows:

Theorem 1. $\varkappa(M) - \varkappa(L) = \varkappa(A)$.

Thus the question under consideration reduces to the following: when is $\varkappa(A) = 0$? Moreover, Theorem 1 shows that the difference $\varkappa(M) - \varkappa(L)$ does not depend on the boundary condition (3).

Let us now consider an arbitrary singular integral operator A of the form (5), possessing the properties listed after formula (5). Its symbol $\sigma(x, \xi)$ is a square matrix of order N , consisting of functions infinitely differentiable for $\xi \neq 0$, homogeneous in ξ of degree 0. For sufficiently large $|x|$ the symbol σ coincides with E . We shall assume that $\det \sigma \neq 0$ for $\xi \neq 0$. We shall say that the symbol σ

homotopic to E , if there exists a matrix $\sigma_t(x, \xi)$ ($0 \leq t \leq 1$), nonsingular and continuous in the aggregate of the variables for $\xi \neq 0$, homogeneous of degree 0 in ξ , and coinciding with σ for $t = 0$ and with E for $t = 1$ and for sufficiently large $|x|$.

Lemma 3. Each of the following conditions is sufficient for the index of the operator A to be equal to zero:

- I. The symbol σ of the operator A is homotopic to E .
- II. $n > N$.
- III. n is odd and $\sigma(x, -\xi) = c(x)\sigma(x, \xi)$, where $c(x)$ is a matrix independent of ξ .

The sufficiency of condition I follows from the possibility of approximating a continuous function by an infinitely differentiable one and from the fact that the norm of the difference of two singular integral operators can be estimated by the difference of their symbols ⁽⁹⁾.

The sufficiency of conditions II and III can be derived from results recently obtained by A. I. Volpert and A. S. Dynin for singular integral operators on closed manifolds. For this purpose, by means of stereographic projection, the operator A must be transformed into a singular integral operator on the n -dimensional sphere, degenerating into multiplication by E in a neighborhood of the north pole of the sphere. The sufficiency of conditions II and III can also be proved directly; this is done more simply than in the general case of A. I. Volpert and A. S. Dynin. Here we shall give a direct proof of the sufficiency of condition II.

Condition I shows that the symbol σ can be regarded as a continuous mapping of the set $E_x^n \times S_\xi^{n-1}$, where E^n is the n -dimensional ball and S^{n-1} the $(n-1)$ -dimensional sphere, into the space of nonsingular matrices of order N , coinciding with the mapping to E for x belonging to the boundary of the ball. Without loss of generality one may assume that σ is a unitary matrix, that $\sigma = E$ for some $\xi = \xi_0$, and that $n = N + 1$. Then σ defines an element of the homotopy group $\pi_{2N+1}(U(N))$. It is known that it is finite (see, for example, ⁽¹⁰⁾). Hence it follows that $\chi(A) = 0$.

For $n \leq N$ the group $\pi_{2n-1}(U(N))$ is a free cyclic group ⁽¹¹⁾. Hence there follows the existence of such a symbol σ_0 (of the type considered by us) that for any other symbol σ , for some integer m , the symbol $\sigma_0^m \sigma^{-1}$ is homotopic to E . If A and A_0 are operators of the form (5) with symbols σ and σ_0 , then $\chi(A) = m\chi(A_0)$. Thus the index $\chi(A)$ may be considered known up to a constant integer factor $\chi(A_0)$. But here an important question remains open: not only do we not know how to compute the index $\chi(A_0)$, we do not even know whether it differs from zero.

From Theorem 1 and Lemma 3 it follows:

Theorem 2. *Each of the following conditions is sufficient for the equality $\chi(L) = \chi(M)$:*

- I. *The symbol $\sigma(x, \xi)$, equal to QP^{-1} for $x \in G$ and to E for $x \notin G$, is homotopic to E .*
- II. *$n > N$.*
- III. *n is odd.*

In conclusion I express my deep gratitude to M. I. Vishik for discussion of a number of questions connected with the index, and also for his attention to the present work.

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