PAIRS OF \(T\)-CONGRUENCES WITH CONSTANT DISTANCE AND CONSTANT ANGLE BETWEEN CORRESPONDING RAYS

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Abstract

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O. S. REDOZUBOVA

PAIRS OF T-CONGRUENCES WITH CONSTANT DISTANCE AND CONSTANT ANGLE BETWEEN CORRESPONDING RAYS

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1. We consider pairs of T-congruences ([3]) with constant distance between corresponding rays and constant angle between them. The vertex O of a rectangular trihedron is placed on a ray of the congruence of common perpendiculars, and the vector $\mathbf{e}_3$ is directed along the ray. The components of the infinitesimal displacement of the trihedron ω_i^j, ω^i are determined by the equations

$$dO = \mathbf{e}_i \omega^i, \qquad d\mathbf{e}_i = \omega_i^j \mathbf{e}_j \qquad (i, j = 1, 2, 3).$$

Here $\omega_i^j = -\omega_j^i, \quad \omega_i^i = 0.$

The pairs T are determined by the system of equations

$$\rho_k H + \frac{\rho_1 \rho_2}{h_1 - h_2}\,\Omega_{k3} - \rho_k A + \frac{\Omega_k^* + h_k \Omega_{k3}^*}{\sin(\alpha_1 - \alpha_2)} = 0,$$

$$\rho_k' H + \frac{\rho_1' \rho_2'}{h_1 - h_2}\,\Omega_{k3} - \rho_k' A + \frac{\Omega_k^* + h_k \Omega_{k3}^*}{\sin(\alpha_1 - \alpha_2)} = 0, \tag{1}$$

$$h_1 - h_2 = \text{const}, \qquad \alpha_1 - \alpha_2 = \text{const} \qquad (k = 1, 2).$$

Here h_k denote the abscissas of the points of intersection of the rays of the pair with the rays of the congruence of common perpendiculars, measured from the point O; α_k are the angles formed by the rays of the pair with the vector $\mathbf{e}_1$; ρ_k, ρ_k' are the abscissas of the foci F_k, F_k' of the congruences of the pair, measured from the points of intersection with the rays of the congruence of common perpendiculars of the corresponding rays of the pair;

$$\Omega_k^* = \omega^1 \sin\alpha_k - \omega^2 \alpha_k, \qquad \Omega_{k3}^* = -\omega_1^3 \sin\alpha_k + \omega_2^3 \cos\alpha_k,$$

$$\Omega_{k3} = \omega_1^3 \cos \alpha_k + \omega_2^3 \sin \alpha_k;$$

$$A = \frac{\omega_1^2 + d\alpha_k}{\sin(\alpha_1 - \alpha_2)}, \qquad H = \frac{\omega^3 + dh_k}{h_1 - h_2}.$$

A sequence of transformations leads to the system of equations

$$A = \alpha\Omega_{13} + \beta\Omega_{23},$$

$$H = -\beta\Omega_{13} - \alpha\Omega_{23},$$

$$\frac{\Omega_1^* + h_1\Omega_{13}^*}{\sin(\alpha_1 - \alpha_2)} = \lambda\Omega_{13} + \mu\Omega_{23}, \tag{2}$$

$$\frac{\Omega_2^* + h_2\Omega_{23}^*}{\sin(\alpha_1 - \alpha_2)} = \mu\Omega_{13} + \lambda\Omega_{23},$$

where $\alpha,\ \beta,\ \lambda,\ \mu$ are functions of the abscissas of the foci and of the distances between the corresponding-

corresponding rays. Investigation of the system of equations makes it possible to conclude:

Pairs T with constant distance and constant angle between corresponding rays exist with an arbitrariness of four functions of one argument.

Since, upon substituting the forms (1) into the equations defining the conditions of layering (see ([3]), p. 403), these equations are satisfied, the *pairs T of congruences with constant distance and constant angle between corresponding rays are layerable.*

Further, the configuration is related to the Gishar trihedral ([2]), whose vertex is located at the center of the ray of the congruence of common perpendiculars, and the vectors $\mathbf{e}_k$ are directed along the bisectors of its focal planes. After substitution into the equations of system (2)

$$\omega^1 = -\hat{\rho}\ \mathrm{ctg}\,\hat{\varphi}\,\omega_2^3, \qquad \omega^2 = -\hat{\rho}\ \mathrm{tg}\,\hat{\varphi}\,\omega_1^3,$$

where $2\hat{\rho}$ is the focal distance, and $2\hat{\varphi}$ is the angle between the focal planes of the congruence of common perpendiculars, the last two equations take the form

$$\begin{aligned} h_1 + h_2 = 0, \qquad \sin(\alpha_1 + \alpha_2) = 0, \\ \frac{2\hat\rho}{\sin 2\hat\varphi} = \frac{2h}{\sin 2\alpha}, \qquad \lambda = \hat\rho \operatorname{tg} \hat\varphi - h \operatorname{tg} \alpha. \end{aligned} \tag{3}$$

Since each pair of corresponding rays is equally inclined to the bisectors of the focal planes and is located at equal distances from the center of the ray of the congruence of common perpendiculars, the *pairs are symmetric. The distance between the boundary points of the rays of the congruence of common perpendiculars is constant.*

2. Special cases are considered.

1) $\mu = 0$. Then either $\rho_1\rho_2 = \rho'_1\rho'_2$, or $\rho_1\rho'_2 = \rho_2\rho'_1$, or $\rho_1\rho_2 = \rho'_1\rho'_2$, $\rho_1\rho'_2 = \rho_2\rho'_1$.

In the first case, from the equations of system (1), if one sets aside the case in which the congruences degenerate into ruled surfaces, it follows that $\rho_1 - \rho'_1 = \rho_2 - \rho'_2$. Consequently, *the pairs are pairs T of the 2nd type.* Obviously, if the pairs are pairs T of the 2nd type, then $\rho_1\rho_2 = \rho'_1\rho'_2$. *In order that pairs T with constant distance and constant angle between corresponding rays be pairs of the 2nd type, it is necessary and sufficient that the abscissas of the foci be inversely proportional.*

In the second case, when $\rho_1\rho'_2 = \rho_2\rho'_1$, comparison of the quadratic equations of system (2) leads, in particular, to the equations

$$[d\lambda, \Omega_{13}] = 0 \qquad [d\lambda, \Omega_{23}] = 0.$$

Hence $\lambda = \text{const}$, and *such pairs, therefore, are defined with an arbitrariness of two functions of one argument.*

The last of equations (3) can be written in another form:

$$\lambda + 2h\sin^2\alpha = 2h\sin^2\hat\varphi.$$

Since the left-hand side of the equation is constant, the right-hand side and $\hat\varphi$ are also constant; moreover, the distance between the boundary points of the congruence of common perpendiculars is constant, and consequently the distance between the foci of this congruence is also constant. *The congruence of common perpendiculars of the pair under consideration is thus pseudospherical.*

It is known that only a pseudospherical congruence possesses a layerable pair of congruences formed by the normals of the focal surfaces ([3]); consequently, among such pairs there are also Ermolaev pairs.

The focal surfaces of the congruence of common perpendiculars will have as their normals the corresponding rays of the pair under the condition that

$$(\mathbf{e}_1 \sin\hat{\varphi} - \mathbf{e}_2 \cos\hat{\varphi}) \times (\mathbf{e}_1 \cos\alpha + \mathbf{e}_2 \sin\alpha) = 0,$$

or under the condition that $\varphi + \alpha = \pi/2$. The pairs under consideration will be Ermolaev pairs; consequently,

$$\lambda = d^2 \cos 2\alpha.$$

In the third case

$$\rho_1\rho_2 = \rho_1'\rho_2', \qquad \rho_1\rho_2' = \rho_2\rho_1'. \tag{4}$$

It follows from equations (1) that the pairs are simultaneously pairs of the 1st and 2nd types. Since for T-pairs of the 1st and 2nd types conditions (4) are satisfied, in order that T-pairs with constant distance and constant angle between corresponding rays be simultaneously pairs of the 1st and 2nd types, it is necessary and sufficient that the rays of the congruence of common perpendiculars intersect the corresponding rays of the pair at the centers ($\rho_1 = -\rho_1',\ \rho_2 = -\rho_2'$). The equations determining such pairs have the form

$$H = A, \qquad \frac{\Omega_k^* + h_k\Omega_{k3}^*}{\sin(\alpha_1 - \alpha_2)} = -\Omega_{k3}\frac{(\rho_1)^2}{h_1 - h_2}.$$

Of the quadratic equations, the first is satisfied identically, while the second and third coincide and reduce to the form

$$[\Omega_{13} + \Omega_{23},\, H] = 0.$$

Such pairs exist with an arbitrariness of one function of one argument.

2) If, for T-pairs with constant distance and constant angle between corresponding rays, the additional congruences are normal, then $\rho_1\rho_2 = \rho_1'\rho_2'$, and, consequently, the pairs are T-pairs of the 2nd type. It can be proved that the pairs exist with an arbitrariness of two functions of one argument.

Moscow State Pedagogical Institute
named after V. I. Lenin

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Note: Figure translations are in progress. See original paper for figures.