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Abstract

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ON THE COMPUTATION OF THE POINCARÉ INDEX

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1. Suppose that in a neighborhood of the zero point of the plane with coordinates x, y a continuous vector field is given

$$\Phi(x,y) = \{\varphi(x,y), \psi(x,y)\}. \tag{1}$$

and $\Phi(0,0) = 0$, i.e. the zero point is singular.

In various questions of function theory, the qualitative theory of differential equations, and functional analysis, one has to answer the question whether the zero singular point of the field (1) is isolated and what its index is (see, for example, ([1–8])). Recall that a zero singular point is called isolated if, for small positive $x^2 + y^2$, the vectors of the field (1) are nonzero; the index (Poincaré index) of an isolated singular point is the angle of rotation of the vector $\Phi(x,y)$, divided by 2π, when one traverses counterclockwise circles of small radius centered at the singular point.

Methods are known for computing the index in the principal particular cases. In the present paper an algorithm is given for computing the index for a case which is, in a certain sense, general. At the same time this algorithm gives a series of sufficient conditions for the isolation of the zero singular point of the field (1). The study of the integral expressing the index ([1]) seems to us difficult in the general case.

We shall assume that the field (1) admits the representation

$$\Phi(x,y) = C_1(x,y) + \cdots + C_M(x,y) + \omega(x,y), \tag{2}$$

where $C_i(x,y)$ are homogeneous operators of order i, whose components are polynomials, and $\omega(x,y)$ is infinitesimal of higher order than $(|x| + |y|)^M$.

Let us denote by $\Phi_i(x,y)$ the vector field

$$\Phi(x,y) = C_1(x,y) + \dots + C_i(x,y). \tag{3}$$

We shall call the field (3) nondegenerate if, for some $\rho > 0$, the inequality

$$\|\Phi_i(x,y)\| \geq \alpha(|x| + |y|)^i \quad (\alpha > 0,\ |x| + |y| \leq \rho) \tag{4}$$

holds.

It can be shown that the field (3) is nondegenerate if and only if the zero singular point of each vector field $\psi(x,y) = \Phi_i(x,y) + \bar{\omega}(x,y)$ (in particular, of the field (1)) is isolated under the condition that $\|\bar{\omega}(x,y)\| = O(|x| + |y|)^i$. In this case the index of the zero isolated singular point of the field $\psi(x,y)$ is equal to the index of the zero singular point of the field (3).

The algorithm described below, as a result of a finite number of steps, indicates the least number m such that for $i = m$ the field (3) is nondegenerate, or shows that the field (3) is degenerate for all $i \leq M$. At the same time, a method is indicated for computing the index of the zero singular point of the nondegenerate field (3). Thus, the proposed algorithm makes it possible to compute the index of the zero singular point of the field (1) in all those cases in which it is determined by a finite number of terms in the Taylor expansions of the functions $\varphi(x,y)$ and $\psi(x,y)$.

2. Choose the number m_0 so that the condition

$$C_1(x,y) \equiv \dots \equiv C_{m_0-1}(x,y) \equiv 0, \qquad C_{m_0}(x,y) \not\equiv 0. \tag{5}$$

is satisfied. Since the field $\Phi_{m_0}(x,y) = C_{m_0}(x,y)$ is homogeneous, for its nondegeneracy it is necessary and sufficient that it have no singular points other than the zero point.

Let

$$C_{m_0}(x,y) = \{a_0(x,y),\ b_0(x,y)\}. \tag{6}$$

The nondegeneracy of the field $\Phi_{m_0}(x,y)$ means that the polynomials $T_0(k) = a_0(1,k)$ and $T_1(k) = b_0(1,k)$ have no common real roots and that the degree of at least one of them is equal to m_0. Suppose that the polynomial $T_0(k)$ has degree m_0. Using Euclid' s algorithm, we construct a sequence of polynomials (the generalized Sturm sequence)

$$T_0(k),\ T_1(k), \dots, T_{l_0}(k), \tag{7}$$

where $T_{i-1}(k) = a_i(k)T_i(k) - T_{i+1}(k)$. By $s(k_0)$ we denote the number of sign changes in the sequence (7) for $k = k_0$; the value of $s(k)$ for sufficiently large positive k will be denoted by $s(+\infty)$; $s(-\infty)$ is defined analogously.

Theorem 1. *Let the field $\Phi_{m_0}(x,y)$ be nondegenerate. Then the index γ of its zero singular point is expressed by the formula*

$$\gamma = s(+\infty) - s(-\infty). \tag{8}$$

Theorem 1 in an equivalent form was already known to Cauchy (see, for example, (9)).

3. Let now the field $\Phi_{m_0}(x,y)$ be degenerate and let $k_1,\ldots,k_s$ be the common real roots of the polynomials $T_0(k)$ and $T_1(k)$ (if the degrees of both polynomials are less than m_0, then among the roots k_i there is the root $k=\infty$). In this case the vectors of the field $\Phi_{m_0}(x,y)$ vanish on $2s$ rays

$$L_1: \ y=k_1x \ (x\geq 0), \qquad L_2: \ y=k_1x \ (x\leq 0),$$
$$\ldots \tag{9}$$
$$L_{2s-1}: \ y=k_{sx} \ (x\geq 0), \qquad L_{2s}: \ y=k_{sx} \ (x\leq 0)$$

(the root $k=\infty$ corresponds to the rays $x=0,\ y\geq 0$ and $x=0,\ y\leq 0$). The rays (9) will be called the rays of degeneration of the field $\Phi_{m_0}(x,y)$.

The field $\Phi_{m_0}(x,y)$ has the form $\psi_0(x,y)d_0(x,y)$, where $d_0(x,y)$ is the greatest common divisor of the polynomials $a_0(x,y)$ and $b_0(x,y)$, while the homogeneous field $\psi_0(x,y)$ is nondegenerate. Denote by γ_0 the index of the zero singular point of this field. The equality

$$\gamma_0 = s(+\infty) - s(-\infty). \tag{10}$$

holds.

Consider, on the set of points $\Gamma_\sigma(\varepsilon)$ $(\sigma=1,\ldots,2s)$, lying in an angle of 2ε radians whose bisector is the ray L_σ, the vector field $\Phi_m(x,y)$. We shall say that the ray of degeneration L_σ is nonsingular for the field $\Phi_m(x,y)$ if, for some $\varepsilon>0$ and $\rho>0$, on the set $\Gamma_\sigma(\varepsilon)$ the inequality

$$\|\Phi_m(x,y)\| \geq \alpha(|x|+|y|)^m \qquad (\alpha>0,\ |x|+|y|\leq\rho) \tag{11}$$

is satisfied. If the ray L_σ is nonsingular for the field $\Phi_m(x,y)$, then there exists the double limit

$$\gamma_\sigma = \lim_{\varepsilon\to 0}\lim_{\rho\to 0}\gamma_\sigma(\Phi_m,\varepsilon,\rho), \tag{12}$$

where $\gamma_\sigma(\Phi_m,\varepsilon,\rho)$ denotes the rotation of the field $\Phi_m(x,y)$ on the part of the circle S_ρ $(x^2+y^2=\rho^2)$ lying in $\Gamma_\sigma(\varepsilon)$. The number γ_σ is a multiple of one-half and does not depend on m. We shall call it the characteristic of the ray of degeneration L_σ.

Theorem 2. *The field $\Phi_m(x,y)$ is nondegenerate if and only if all the rays of degeneration (9) are nonsingular for it.*

The index γ of the zero isolated singular point of the nondegenerate field $\Phi_m(x,y)$ is determined by the formula

$$\gamma = \gamma_0+\gamma_1+\ldots+\gamma_{2s}. \tag{13}$$

For the application of Theorem 2 it is necessary to know whether the ray of degeneration L_σ is nonsingular for the field $\Phi_m(x,y)$, and, if it is nonsingular, its characteristic γ_σ. Below we indicate one of the methods for investigating the ray of degeneration L_σ.

4. We shall assume that the ray of degeneration L_σ coincides with the positive semiaxis of ordinates $x = 0,\ y \geqslant 0$. This entails no loss of generality, since otherwise one can pass to a new coordinate system.

Denote by $a_i^j,\ b_i^j$ the coefficients of $x^i y^j$ in the expansions in powers of x, y of the functions $\varphi(x, y)$ and $\psi(x, y)$. Let

$$\begin{aligned} a_0(x,y) &= a_{m_0}^0 x^{m_0} + \dots + a_\mu^{m_0-\mu} x^\mu y^{m_0-\mu}, \\ b_0(x,y) &= b_{m_0}^0 x^{m_0} + \dots + b_\mu^{m_0-\mu} x^\mu y^{m_0-\mu}, \end{aligned} \tag{14}$$

where at least one of the numbers $a_\mu^{m_0-\mu},\ b_\mu^{m_0-\mu}$ is different from zero. In the expansions in powers of x, y of the functions $\varphi(x, y)$ and $\psi(x, y)$, consider those terms for which

$$i < \mu, \quad j > m_0 - \mu.$$

We shall call the weight r of such a term the number

$$\frac{j - m_0 + \mu}{\mu - i}.$$

The sum of the terms $a_i^j x^i y^j,\ b_i^j x^i y^j$ of the functions $\varphi(x, y)$ and $\psi(x, y)$, respectively, will be denoted by $c^{(r)}(x, y)$ and $d^{(r)}(x, y)$.

In the case where all polynomials $c^{(r)}(x, y),\ d^{(r)}(x, y)$ for

$$r \leqslant \frac{M - m_0 + \mu}{\mu}$$

are identically equal to zero, the field (3) for any $i \leqslant M$ vanishes on the ray L_σ and, consequently, is degenerate.

Let us now consider the case where, for some

$$r_0 \leqslant \frac{M - m_0 + \mu}{\mu},$$

all polynomials $c^{(r)}(x, y),\ d^{(r)}(x, y)$ are identically equal to zero for $r < r_0$, and at least one of the polynomials $c^{(r_0)}(x, y),\ d^{(r_0)}(x, y)$ assumes nonzero values.

Denote by $\Phi^{(\sigma)}(x, y)$ the vector field

$$\Phi^{(\sigma)}(x, y) = \{a_\sigma(x, y), b_\sigma(x, y)\}, \tag{15}$$

where

$$a_\sigma(x,y) = a_\mu^{m_0-\mu} x^\mu y^{m_0-\mu} + c^{(r_0)}(x,y),$$
$$b_\sigma(x,y) = b_\mu^{m_0-\mu} x^\mu y^{m_0-\mu} + d^{(r_0)}(x,y). \tag{16}$$

The field $\Phi^{(\sigma)}(x,y)$ has the form $\psi^{(\sigma)}(x,y)d_\sigma(x,y)$, where $d_\sigma(x,y)$ is the greatest common divisor of the polynomials (16), and the field $\psi^{(\sigma)}(x,y)$ has, for $y \geqslant 0$, no singular points different from zero. We denote by $\gamma_{\sigma,0}$ the rotation of this field on the semicircle $S_1\{x^2+y^2=1,\ y \geqslant 0\}$. This rotation is readily computed. Suppose, for example, that $a_\mu^{m_0-\mu} \neq 0$ and

$$U_0(k),\ U_1(k), \dots,\ U_{l_\sigma}(k) \tag{17}$$

is the generalized Sturm sequence of the polynomials $U_0(k) = a_\sigma(k,1)$ and $U_1(k) = b_\sigma(k,1)$, and $s_\sigma(k_0)$ is the number of changes of sign in the sequence (17) for $k = k_0$. Then the formula holds

$$\gamma_{\sigma,0} = \frac{s_\sigma(-\infty) - s_\sigma(+\infty)}{2}. \tag{18}$$

We shall say that the field $\Phi^{(\sigma)}(x,y)$ is nondegenerate for $y \geqslant 0$ if it has no singular points for $y > 0$. This means that the polynomials $U_0(k)$ and $U_1(k)$ have no common real roots.

Theorem 3. *Let the field $\Phi^{(\sigma)}(x,y)$ be nondegenerate for $y \geqslant 0$. Then the ray of degeneration L_σ is nonsingular for the field $\Phi_m(x,y)$ for $m \geqslant m_0 + \mu(r_0 - 1)$, and its characteristic γ_σ is determined by the formula*

$$\gamma_\sigma = \gamma_{\sigma,0}. \tag{19}$$

5. We pass to the case where the field $\Phi^{(\sigma)}(x,y)$ is degenerate for $y \geqslant 0$. Make the substitution

$$x = |u|^p \operatorname{sign} u, \qquad y = |v|^q \operatorname{sign} v, \tag{20}$$

where p and q are relatively prime integers determined from the condition $r_0 = p/q$. The resulting field

$$\Phi^*(u,v) = \Phi\big(|u|^p \operatorname{sign} u,\ |v|^q \operatorname{sign} v\big) \tag{21}$$

admits the representation

$$\Phi^*(u,v) = C^*_{m_0 q + \mu(p-q)}(u,v) + \dots + C^*_{Mq}(u,v) + \omega^*(u,v), \tag{22}$$

where $C_i^*(u,v)$ are positively homogeneous operators of order i, with

$$C^*_{m_0q+\mu(p-q)}(u,v) = \Phi^{(\sigma)}(|u|^p \operatorname{sign} u,\ |v|^q \operatorname{sign} v),$$

and $\omega^*(u,v)$ is infinitesimal of higher order than $(|u|+|v|)^{Mq}$.

Denote by $\Phi_i^*(u,v)$ the vector field

$$\Phi_i^*(u,v) = C^*_{m_0q+\mu(p-q)}(u,v) + \cdots + C_i^*(u,v). \tag{23}$$

It can be shown that the ray of degeneracy L_σ will be nonsingular for the field $\Phi_m(u,v)$ if and only if, for every $\varepsilon_1 > 0$, for some $\rho > 0$ the inequality

$$\left\|\Phi^*_{mq}(u,v)\right\| \geq \alpha(|u|+|v|)^{mq} \quad (\alpha > 0,\ u^2+v^2 \leq \rho^2,\ |u| \leq \varepsilon_1 v). \tag{24}$$

holds. The characteristic γ_σ of the nonsingular ray of degeneracy L_σ is then expressed by the formula

$$\gamma_\sigma = \lim_{\varepsilon_1\to 0}\lim_{\rho\to\infty} \gamma(\Phi^*_{mq}, \varepsilon_1, \rho), \tag{25}$$

where by $\gamma(\Phi^*_{mq}, \varepsilon_1, \rho)$ is denoted the rotation of the vector field $\Phi^*_{mq}(u,v)$ on the part of the circle $S_\rho^{\varepsilon_1}(u^2+v^2=\rho^2,\ |u| \leq \varepsilon_1 v)$.

In the case when one of the numbers p,q is even, the field $\Phi^*(u,v)$ is not a field of the type considered above: the components of the operators $C_i^*(u,v)$ will be polynomials whose coefficients will contain the factors $\operatorname{sign} u$ (p even) or $\operatorname{sign} v$ (q even). However, all the methods set forth above carry over also to fields of this class.

The components of the field $C^*_{m_0q+\mu(p-q)}(u,v)$ vanish on several rays

$$L_{\sigma,1}:\ u = k_1 v\ (v \geq 0), \dots, L_{\sigma,s_1}:\ u = k_{s_1} v\ (v \geq 0), \tag{26}$$

where k_i are the common real roots of the equations $a_\sigma(|k|^p \operatorname{sign} k, 1) = 0$ and $b_\sigma(|k|^p \operatorname{sign} k, 1) = 0$ (the number of these roots does not exceed μ). The rays (26) will be called rays of degeneracy of the field $C^*_{m_0q+\mu(p-q)}(u,v)$. For each ray of degeneracy $L_{\sigma,\sigma'}$, nonsingular for some field $\Phi_i^*(u,v)$, one may introduce the notion of the characteristic $\gamma_{\sigma,\sigma'}$.

Theorem 4. *The ray of degeneracy L_σ will be nonsingular for the field $\Phi_m(x,y)$ if and only if all rays of degeneracy (26) are nonsingular for the field $\Phi^*_{mq}(u,v)$.*

The characteristic γ_σ of the nonsingular ray of degeneracy L_σ is then determined by the formula

$$\gamma_\sigma = \gamma_{\sigma,0} + \gamma_{\sigma,1} + \cdots + \gamma_{\sigma,s_1}. \tag{27}$$

6. To the rays of degeneracy (26) one may again apply the scheme set forth in §§ 4–5, and so on. The application of Theorems 3 and 4 constitutes the proposed algorithm for studying the ray of degeneracy L_σ.

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CITED LITERATURE

1. A. Poincaré, *On curves defined by differential equations*, 1947.
2. C. Lefschetz, *Geometric Theory of Differential Equations*, IL, 1960.
3. V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, 1949.
4. J. Leray, J. Schauder, UMN, 1, no. 3–4 (1946).
5. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
6. V. B. Melamed, DAN, 126, no. 1 (1959).
7. P. P. Zabreiko, M. A. Krasnosel' skii, DAN, 141, no. 2 (1961).*
8. P. B. Cherevichnyi, Tr. Odessa Polytechnical Institute, 21, 23 (1960).
9. D. A. Grave, *Elements of Higher Algebra*, Kiev, 1914.

* In paper (7), on p. 292, in formula (4), in the right-hand side of the inequality, $\|P_0x\|^2$ is printed; it should be $\|P_0x\|^r$.

Note: Figure translations are in progress. See original paper for figures.