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Abstract

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ON THE CLASS NUMBER

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In connection with A. Weil' s theorem ([1]) on the zeros of the congruence zeta-function $Z(s)$ for a curve over a finite field, Brauer' s theorem ([2]), which generalizes Siegel' s theorem to arbitrary number fields, and Fogels' theorem ([3]) on the boundary of the zeros for Hecke L-series, it has become possible to study segments of Euler products of the corresponding zeta-functions and L-series in relation to the boundary of their zeros. In what follows, by $\zeta_K(s)$ we shall mean the Dedekind zeta-function of an algebraic number field K over the field of rational numbers, $L_K(s,\chi)$ the Hecke or Artin series of this field, and $Z(s)$ the congruence zeta-function for a curve over a finite field of characteristic p.

Theorem 1 of the paper ([4]) admits a generalization to algebraic fields, and by analogy one may obtain the relation

$$\prod_{N(\mathfrak{p})\leq D}\left(1-\frac{1}{N(\mathfrak{p})^s}\right)\zeta_K(s)(s-1)=\frac{e^{-c}}{\ln D}\,F(s,D), \tag{1}$$

where $F(s,D)$ is a certain analytic function depending on the boundary of the zeros of $\zeta_K(s)$. But if for the rational field, as $s\to 1$, one obtains the limiting equality

$$\lim_{s\to 1}\zeta_K(s)(s-1)=1,$$

from which one can obtain some information about the segment of the Euler product at the point 1, then in the case of an arbitrary algebraic field

$$\lim_{s\to 1}\zeta_K(s)(s-1)=\mu H,$$

where

$$\mu=\frac{2^{r_1+r_2}\pi^{r_2}}{\omega}\,\frac{R}{\sqrt{|d|}};$$

d is the discriminant of the field; R is the regulator; $n = r_1 + 2r_2$ is the degree of the field; ω is the number of principal roots of unity, and in the limiting relation obtained from (1) as $s \to 1$, the class-number function H comes to the fore, since the segment of the Euler product

$$1 < \prod_{N(\mathfrak{p}) \le D} \left(1 - \frac{1}{N(\mathfrak{p})}\right)^{-1} < (\ln D)^n$$

is considerably simpler in its structure and has a good trivial upper and lower estimate under the condition that

$$\ln D = c(n) \ln |d| \, \ln \ln |d|.$$

A detailed proof of equality (1) leads to the following formulas for the class number.

If $\zeta_K(s)$ has no Siegel zero, then

$$H = \frac{\omega}{2^{r_1+r_2} \pi^{r_2}} \frac{\sqrt{|d|}}{R} \frac{e^{-c}}{\ln D} \prod_{N(\mathfrak{p}) \le D} \left(1 - \frac{1}{N(\mathfrak{p})}\right)^{-1} \left(1 + \frac{\theta}{\ln D}\right), \tag{2}$$

where c is Euler' s constant.

If, however, $\zeta_K(s)$ has a real Siegel zero γ, then

$$H = \frac{\omega}{2^{r_1+r_2} \pi^{r_2}} \frac{\sqrt{|d|}}{R} (1-\gamma) e^{-\omega_1(D)} \prod_{N(\mathfrak{p}) \le D} \left(1 - \frac{1}{N(\mathfrak{p})}\right)^{-1} \left(1 + \frac{\theta}{\ln D}\right) \tag{3}$$

under the condition that $\ln D \ge c(n) \ln |d| \ln \ln |d|$, and the function $\omega_1(D)$, for $\ln D = c(n) \ln |d| \ln \ln |d|$, does not exceed a certain absolute constant. As $D \to \infty$,

$$(1-\gamma) e^{-\omega_1(D)} \prod_{N(\mathfrak{p}) \le D} \left(1 - \frac{1}{N(\mathfrak{p})}\right)^{-1} \to \chi_K,$$

where χ_K is a certain constant of the field K. In particular, if K is a quadratic field, then

$$\chi_K = L(1, \chi),$$

and we obtain the known formula for a quadratic field. From relations (2) and (3) one obtains the asymptotic relations, as $|d| \to \infty$:

$$\ln HR = \ln \sqrt{|d|} + O(\ln \ln |d|),$$

and in the case of a Siegel zero:

$$\ln HR = \ln \sqrt{|d|} + \ln(1-\gamma) + O(\ln \ln |d|).$$

Let us note that a relation of type (1) for the congruence zeta-function $Z(s)$ already assumes the limiting form in the sense of the behavior of the function $E(s, D)$, by A. Weil' s theorem on the distribution of the zeros of this function on $\operatorname{Re} s = \frac{1}{2}$.

If the norm of a prime divisor is denoted by $|\mathfrak{p}|$, and g is the genus of the field, then, analogously to (2), from a formula of type (1) one can obtain:

$$h = p^g \frac{e^{-c-\omega}}{\ln D} \prod_{|\mathfrak{p}| \le D} \left(1 - \frac{1}{|\mathfrak{p}|}\right)^{-1} \exp\left(\frac{\theta g \ln D}{\sqrt{D}}\right), \tag{4}$$

where h is the number of divisor classes, $D = p^{f+1/2}$, $f = 1, 2, 3, ...,$ $|\theta| \le 2$. If we put $D = p^{3/2}$, then from (4) we obtain

$$h = \chi_0 p^g \exp\left(\frac{\theta g}{\sqrt{p}}\right),$$

where χ_0 is an absolute positive constant. Let us note that this relation for h was known earlier (see, for example, ([5]), p. 140). Here another, analytic proof of this fact is given.

Segments of Euler products of Artin and Hecke L-series are studied similarly. For them one can obtain the relation:

$$\prod_{N(\mathfrak{p}) \le D} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right) \frac{L_K(s, \chi)}{s - \gamma} = F_K(s, D); \tag{5}$$

χ is a real character, γ is a Siegel zero. From this relation and Brauer' s theorem (2), it is already not difficult to obtain an estimate for $(1-\gamma)$:

$$1 - \gamma > \frac{c(\varepsilon)}{|dq^{n/2}|^{\varepsilon}}, \tag{6}$$

where $\varepsilon > 0$ is any positive number; $c(\varepsilon) > 0$ is a constant depending only on ε; q is the modulus of the real character χ; n is the degree of the field K; d is the discriminant. It is obtained according to the following scheme: we multiply

$\zeta_K(s)$ and $L_K(s,\chi)$. This corresponds to a quadratic extension of the field K. Denote the resulting field by K_1. But, by Artin' s theorem,

$$\zeta_{K_1}(s) = \zeta_K(s) L_K(s,\chi). \tag{7}$$

The zero of the zeta-function is simple, and we assumed that it occurs in the expansion of $L_K(s,\chi)$; therefore, multiplying the right- and left-hand sides of (7) by $(s-1)$ and letting $s \to 1$, we obtain

$$\mu H L_K(s,\chi) = \mu_1 H_1.$$

Applying to this relation Brauer' s theorem [(2)], equality (5), and the theorem on the discriminant of the extended field, we obtain the estimate (6).

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REFERENCES

[1] A. Weil, *Sur les courbes algébriques et les variétés qui s' en déduisent*, Paris, 1948.
[2] R. Brauer, *Am. J. Math.*, **69**, 2, 243 (1947).
[3] E. Fogels, *Acta Arithm.*, **7**, No. 3 (1962).
[4] S. Lang, *Abelian varieties*, N. Y., No. 7, 1959.
[5] A. I. Vinogradov, *Dokl. Akad. Nauk SSSR*, **143**, No. 5 (1962).

Note: Figure translations are in progress. See original paper for figures.