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Abstract

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SEMIGROUPS WHOSE LATTICE OF SUBSEMIGROUPS HAS RELATIVE COMPLEMENTS

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By $\Sigma'(\Gamma)$, as in (^{1}, ^{2}), we denote the partially ordered, by inclusion, set of all subsemigroups of the semigroup Γ with the empty set adjoined. In other words, $\Sigma'(\Gamma)$ is the partially ordered set of all subsemigroups of the semigroup Γ, if the empty set is also regarded as a subsemigroup. $\Sigma'(\Gamma)$ is a complete lattice.

The present note is devoted to describing semigroups Γ for which $\Sigma'(\Gamma)$ is a lattice with relative complements, i.e. every closed interval is a complemented lattice∗.

For convenience of notation we do not distinguish between an element of a semigroup and the subset consisting of this element; this will not lead to any misunderstanding. However, for the one-element semigroup generated by an idempotent x, we use the usual notation $\{x\}$ (and, in general, by $\{M\}$ we denote the subsemigroup generated by the set M of elements of the given semigroup). By $\cap$ and $\cup$ we denote the set-theoretic operations of intersection and union.

Lemma 1. *In order that the lattice of all subsemigroups of a semigroup be a lattice with relative complements, it is necessary and sufficient that for every triple of subsemigroups A, B, H, where $A \subset H \subset B$, there exist a subsemigroup C such that $A \subset C \subset B$ and $C \cap H = A$**.

Proof. Necessity is trivial. Let us prove sufficiency. Let $A \subset H \subset B$. We shall show that in the closed interval $[A, B]$, H has a complement. By hypothesis, in this interval there exists a subsemigroup distinct from A whose intersection with H is equal to A. But then there also exists a maximal subsemigroup with this property. Let C be such a subsemigroup. Suppose that $\{H, C\} \neq B$, and consider the subsemigroups $C, \{H, C\}, B$. We have $C \subset \{H, C\} \subset B$; therefore, by the hypothesis, it follows that there exists a subsemigroup C_1 such that $C \subset C_1 \subset B$ and $C_1 \cap \{H, C\} = C$. Then

$$C_1 \cap H = C_1 \cap \{H, C\} \cap H = C \cap H = A,$$

and this contradicts the choice of C. Consequently, $\{H, C\} = B$, i.e. C is a complement of H in the interval $[A, B]$. The lemma is proved.

The main content of the note is the following

Theorem 1. *In order that $\Sigma'(\Gamma)$ be a lattice with relative complements, it is necessary and sufficient that all elements of the semigroup Γ be idempotents and that, for any $x, y \in \Gamma$, the alternative*

$$xyx = x, \qquad xyx = y. \tag{*}$$

hold.

$*$ The basic lattice-theoretic concepts are assumed known. For the corresponding definitions, see (^{3}).

$**$ It is easy to see that the assertion of the lemma, formulated here for the lattice of subsemigroups, is also valid for the lattice of subalgebras of any abstract algebra.

Proof. Necessity. Let x be an arbitrary element of the semigroup Γ, for which $\Sigma'(\Gamma)$ is a lattice with relative complements. Consider the closed interval $[\emptyset, \{x\}]$. If $\{x\}$ consists of more than one element, then in $\{x\}$ there exists a proper subsemigroup having nonempty intersection with every other subsemigroup. Indeed, if $\{x\}$ is infinite, then such a subsemigroup is $\{x^2, x^3, ...\}$; if $\{x\}$ is finite, then it contains an idempotent, and the one-element subsemigroup generated by it has the indicated property. In both cases the interval $[\emptyset, \{x\}]$ will not be a lattice with complements. Consequently, $\{x\}$ consists of one element, i.e., x is an idempotent.

Let x, y be two arbitrary distinct elements of the semigroup Γ. Two cases are possible:

$$1) \quad xy = yx;$$
$$2) \quad xy \neq yx.$$

Consider the first of them. If $xy \neq x$ and $xy \neq y$, then in the interval $[\emptyset, \{x, y\}]$ the one-element subsemigroup $\{xy\}$ has no complement, since $\{x, y\} = x \cup y \cup xy$ and $\{x, xy\} = x \cup xy$, $\{y, xy\} = y \cup xy$, while only the subsemigroups $\{x\}$ and $\{y\}$ have empty intersection with $\{xy\}$. Thus in this case $xy \in x \cup y$, and therefore $xyx \in x \cup y$, which means precisely that condition $(*)$ is fulfilled.

We pass to the second case, $xy \neq yx$. We shall show that here necessarily $xyx = x$. Every element of the subsemigroup $\{x, y\}$ is, as is easy to see, one of the following: x, y, xy, yx, xyx, yxy. Suppose that $xyx \neq x$. Then $xy \neq x$, $yx \neq x$. Since $xy \neq yx$, at least one of these elements is not equal to xyx. Let $xy \neq xyx$ (the case $yx \neq xyx$ is completely analogous). Consider the subsemigroup $\{x, xy\}$. It consists of the elements x, xy, xyx. In the interval $[\emptyset, \{x, xy\}]$ the one-element subsemigroup $\{xyx\}$ has no complement, since $\{x, xyx\} = x \cup xyx$, $\{xy, xyx\} = xy \cup xyx$, while only the subsemigroups $\{x\}$ and $\{xy\}$ have empty intersection with $\{xyx\}$. The contradiction obtained proves that $xyx = x$. Note that in the proof of necessity we actually used the weaker condition: every closed interval of the form $[\emptyset, H]$ is a lattice with complements.

For the proof of sufficiency we shall need the following

Lemma 2. Let Γ be a semigroup of idempotents in which condition $(*)$ is fulfilled. Then for any triple x, y, z of elements of Γ one has

$$xyz \in xy \cup xz \cup yz.$$

Proof. Let $zxz = z$. Then $xyz = xyzxz = xz$, if $x \cdot yz \cdot x = x$, and $xyz = xyzxz = yzz = yz$, if $x \cdot yz \cdot x = yz$. Let $zxz = x$. Then $xz = zx = x$ and $xyz = xzyz = xz$, if $zyz = z$, and $xyz = xzyz = xy$, if $zyz = y$. The lemma is proved.

Corollary. In a semigroup satisfying the conditions of the lemma, every element $x_1 x_2 \ldots x_n$ is equal to at least one of $x_i x_j$, $1 \leq i \leq j \leq n$.

We proceed to the proof of **sufficiency**. Let Γ be a semigroup of idempotents in which condition $(*)$ is fulfilled, and let A, B, H be distinct subsemigroups of the semigroup Γ such that $A \subset H \subset B$. Taking Lemma 1 into account, it is enough to show that there is a subsemigroup C such that $A \subset C \subset B$ and $C \cap H = A$*. Suppose the contrary. Then for an arbitrary element $b \in B \setminus H$ we have $\{A, b\} \cap H \neq A$. Denote $D = \{A, b\} \cap H$; $D \setminus A = \emptyset$. Let d be an arbitrary element of $D \setminus A$; d is equal to some product composed of elements of A and the element b. But by virtue of the corollary to Lemma 2, each such product has the form ab or ba, where a is some element of A. Let, for example, $d = ab$ (the case $d = ba$ is completely analogous). Then $ba \in H$. Indeed, by virtue of condition $(*)$ $baab = bab = b$, since from $bab = a$ it would follow that $ab = a$, i.e. $d \in A$, which is contradictory. Therefore, if $ba \in H$, then $b = baab \in H$, and this contradicts the condition $b \in B \setminus H$.

For the element ba let us carry out arguments analogous to those just given for the element b. We obtain that there exists an element $a_1 \in A$ such that $baa_1 \in H \setminus A$ or $a_1 ba \in H \setminus A$. Consider each of these cases.

1. $baa_1 \in H \setminus A$.

We have $baa_1 \neq ba$, since $ba \in H$, and $baa_1 \neq aa_1$, since $aa_1 \in A$. Therefore, by Lemma 2, $baa_1 = ba_1$. Since $ab \in H$ and $ba_1 \in H$, we have $ba_1 ab \in H$, and therefore $ba_1 ab \neq b$, whence, by condition $(*)$, $ba_1 ab = a_1 a$. Substituting

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* Such a subsemigroup always exists if $A = \emptyset$. Namely, as it one may take the one-element subsemigroup generated by an arbitrary element of the set $B \setminus H$. Therefore in the subsequent arguments it is everywhere assumed that $A \neq \emptyset$.

$ba_1 = baa_1$ in the last equality, we obtain $baa_1ab = a_1a$. If $aa_1a = a$, then $baa_1ab = bab$; if $aa_1a = a_1$, then $baa_1ab = ba_1b$. Each of these elements is equal to b, since if, for example, $bab = a$, then $ab = a$, which is impossible, for $ab \in A$. Thus we obtain $a_1a = b$, a contradiction.

2. $a_1ba \in H \setminus A$.

We have $a_1ba \neq a_1a$, since $a_1a \in A$, and $a_1ba \neq ba$, since $ba \in H$. Therefore, by Lemma 2, $a_1ba = a_1b$. Hence, multiplying on the left by b, we obtain $ba_1ba = ba_1b$. As noted in the preceding case, $ba_1b = b$; therefore from the last equality it follows that $ba = b$, whence, in turn, $aba = ab$. But $aba = a$ or $aba = b$, whereas $ab \neq a$ and $ab \neq b$ (indeed $ab \in H \setminus A$). We have obtained a contradiction.

The contradiction obtained in each of the two possible cases shows that there exists a subsemigroup C such that $A \subset C \subset B$ and $C \cap H = A$. It remains to apply Lemma 1.

The theorem is completely proved.

Remark. It is not hard to verify that in a semigroup of idempotents condition $(*)$ is equivalent to the assertion that for any pair of elements x, y one of the following conditions is satisfied:

1) $xy = yx = x$ (or $xy = yx = y$);

2) $xy = x,\ yx = y$;

3) $xy = y,\ yx = x$;

4) $x \neq xy \neq y,\ x \neq yx \neq y,\ xyx = x,\ yxy = y$.

This remark makes it possible to visualize better the multiplication law of the elements of the semigroup under consideration.

Let us recall one definition. A semigroup Γ is called a **band** (4) of its subsemigroups $\Gamma_\alpha, \Gamma_\beta, \ldots$, called the **components** of the band, if all the components are pairwise disjoint, their set-theoretic sum is equal to Γ, and for each pair Γ_ξ, Γ_η of these subsemigroups there is a subsemigroup Γ_σ such that $\Gamma_\xi \Gamma_\eta \subseteq \Gamma_\sigma$. The band is called **strong** (2) if for arbitrary subsemigroups H_α and H_β from different components their composite coincides with the set-theoretic sum, $\{H_\alpha, H_\beta\} = H_\alpha \cup H_\beta$.

In particular, a strong band of one-element subsemigroups is a semigroup all of whose elements are idempotents, and for any pair of its elements their product is equal to one of them.

Theorem 2. *In order that $\Sigma'(\Gamma)$ be a Dedekind structure with complements, it is necessary and sufficient that the semigroup Γ be a strong band of one-element subsemigroups. In this case $\Sigma'(\Gamma)$ is even a Boolean algebra.*

Proof. Necessity. A Dedekind structure with complements is a structure with relative complements (see (^{3})); therefore a semigroup Γ satisfying the condition of the theorem will be a semigroup of idempotents in which condition $(*)$ is fulfilled (see Theorem 1).

Let us show that condition 4) from the remarks to Theorem 1 is impossible in our case. Suppose the contrary: there are elements x and y such that

$x \neq xy \neq y,\ x \neq yx \neq y,\ xyx = x,\ yxy = y$. Denote $X = \{x\},\ Y = \{y\},\ Z = \{x, xy\}$. Then $\{X, Y\} \cap Z = Z$, but $\{X, Y \cap Z\} = X$, i.e. $\{X, Y\} \cap Z \neq \{X, Y \cap Z\}$, which contradicts the Dedekind property of $\Sigma'(\Gamma)$.

Thus, only the cases 1)–3) listed in the remark to Theorem 1 are possible, and this means precisely that Γ is a strong union of one-element semigroups.

Sufficiency. If the semigroup Γ is a strong union of one-element semigroups, then it is easy to see that $\Sigma'(\Gamma)$ coincides with the structure of all subsets of the set Γ, i.e., in particular, is a Boolean algebra.

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