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**Abstract**

**Full Text**

## **Reports of the Academy of Sciences of the USSR**

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**MATHEMATICS**

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### **ON CONDITIONS FOR ORTHOGONALITY AND EQUIVALENCE OF GAUSSIAN MEASURES IN FUNCTION SPACE**

*(Presented by Academician A. N. Kolmogorov on 10 IX 1962)*

It is well known <sup>(1,2)</sup> that Gaussian measures in the function space of realizations corresponding to two different random processes may be either equivalent or orthogonal. However, the necessary and sufficient conditions for equivalence of Gaussian measures <sup>(1-4)</sup> available so far are all insufficiently effective in the sense that verifying them for concrete processes is an extremely difficult and as yet unresolved problem. Therefore, more particular sufficient conditions for orthogonality and equivalence of measures acquire great importance; a number of works <sup>(5-10)</sup> are devoted to finding such conditions. In the present note some new sufficient conditions are introduced for the orthogonality and equivalence of measures corresponding to intervals of real random processes with stationary Gaussian increments and mean 0.

All processes are assumed to be defined on the interval  $0 \leq t \leq 1$ , but the assertions formulated below concerning the orthogonality or equivalence of measures evidently remain valid also for processes defined on any finite interval  $[0, T]$ . We shall assume that all processes considered below have spectral densities. (For the definition of the spectral density of a process with stationary increments, see, for example, <sup>(7)</sup>.) We shall denote the spectral density of the process  $\xi(t)$  by  $f_\xi(t)$ , and the corresponding Gaussian measure in function space by  $m_\xi$ . All results of the present note are formulated in terms of the spectral densities of the processes. Orthogonality of measures will be denoted by the symbol  $\perp$ , equivalence by the symbol  $\sim$ .

Let us denote by  $\xi(t)$  and  $\eta(t)$  two processes with stationary Gaussian increments and mean 0 such that the spectral density of the process  $\eta(t)$  can be represented in the form  $f_\eta(\lambda) = f_\xi(\lambda) + f_\zeta(\lambda)$  (i.e.,  $f_\eta(\lambda) \geq f_\xi(\lambda)$ ).

Then the following theorems hold:

**Theorem 1.** Let the processes  $\xi(t)$  and  $\eta(t)$  be separable, the probability distributions of  $\xi(0)$  and  $\eta(0)$  equivalent,

$$f_{\xi}(\lambda) \geq \frac{c_1}{1 + |\lambda|^{\alpha_1}} \quad (c_1 > 0) \quad (1)$$

and, for some  $\lambda_0 > 0$ ,

$$f_{\zeta}(\lambda) \leq \frac{c_2}{|\lambda|^{\alpha_2}} \quad \text{for } |\lambda| > \lambda_0, \quad (2)$$

where  $\alpha = \alpha_2 - \alpha_1 > \frac{1}{2}$ . Then  $m_{\eta} \sim m_{\xi}$ .

**Theorem 2.** Suppose that, for some  $\lambda_0 > 0$ ,  $c_1 \geq 0$  and  $c_2 > 0$ ,

$$f_{\xi}(\lambda) \leq \frac{c_1}{|\lambda|^{\alpha_1}} \quad \text{for } |\lambda| > \lambda_0, \quad (3)$$

$$f_{\zeta}(\lambda) \geq \frac{c_2}{|\lambda|^{\alpha_2}} \quad \text{for } |\lambda| > \lambda_0, \quad (4)$$

where  $1 < \alpha_1 = \alpha_2 - \frac{1}{2}$ . Then  $m_{\eta} \perp m_{\xi}$ .

Let us briefly outline the proof of these theorems. Consider partitions of the interval  $[0, 1]$  into  $n$  equal subintervals  $\Delta t_j$  ( $j = 1, 2, \dots, n$ ). Let  $\mathbf{x}_n$  be the  $n$ -dimensional column vector of increments, on the intervals  $\Delta t_j$ , of the process specified on  $0 \leq t \leq 1$ . Denote by  $p_{\xi}(\mathbf{x}_n)$  the  $n$ -dimensional density of the normal probability distribution of the vector  $\mathbf{x}_n$  corresponding to the process  $\xi(t)$ , and by  $A_n(\xi)$  the covariance matrix of successive increments of the process  $\xi(t)$  on the intervals  $\Delta t_j$ . The symbol  $\mathbf{M}_{\xi}$  will denote expectation with respect to the distribution corresponding to the process  $\xi(t)$ . Following Gacki <sup>(1)</sup>, put

$$J_n(\xi, \eta) = \mathbf{M}_{\xi} \ln \frac{p_{\xi}(\mathbf{x}_n)}{p_{\eta}(\mathbf{x}_n)} - \mathbf{M}_{\eta} \ln \frac{p_{\xi}(\mathbf{x}_n)}{p_{\eta}(\mathbf{x}_n)}. \quad (5)$$

Then, from the assumption of normality of the finite-dimensional distributions of the processes, the formula is easily derived (cf. <sup>(10)</sup>):

$$\begin{aligned} J_n(\xi, \eta) &= \frac{1}{2} \text{Sp} [A_n(\eta)A_n^{-1}(\xi) - 2E_n + A_n(\xi)A_n^{-1}(\eta)] = \\ &= \frac{1}{2} \text{Sp} \frac{[A_n(\xi)A_n^{-1}(\xi)]^2}{E_n + A_n(\xi)A_n^{-1}(\xi)}, \end{aligned} \quad (6)$$

where  $\text{Sp } A$  is the trace of the matrix  $A$ , and  $E_n$  is the identity matrix of order  $n$ . It is easy to see that it is sufficient to prove our theorems for the case  $1 < \alpha_1 \leq 3$ .

The validity of Theorem 1 for  $\alpha_1 = 2$  follows, for example, from Theorem 10.5.1 of Pinsker's work <sup>(11)</sup>. In the case  $1 < \alpha_1 < 2$ , without loss of generality, we put  $f_\xi(\lambda) = c_1|\lambda|^{-\alpha_1}$  and  $f_\zeta(\lambda) = c_2|\lambda|^{-\alpha_2}$ . Further, using the theorem on Toeplitz forms <sup>(12)</sup>, it can be shown that the eigenvalues of the matrix  $A_n(\zeta)A_n^{-1}(\xi)$  do not exceed the corresponding eigenvalues of the same matrix for  $\alpha_1 = 2$  and the same  $\alpha$ . Hence, using formula (6) and Gacki's criterion <sup>(1)</sup> for the equivalence of Gaussian measures, we find that  $m_\eta \sim m_\xi$ .

In the same case  $2 < \alpha_1 \leq 3$ , Theorem 1 is proved by analogous arguments, using the validity of the theorem for  $3 < \alpha_1 \leq 4$ , which follows from the assertion already proved, and from the fact that multiplying both spectra by  $\lambda^{-2}$  cannot destroy equivalence.

We note that for  $\alpha > 1$  and any  $\alpha_1$  the assertion of Theorem 1 also follows from a recent result of Parzen <sup>(10)</sup>.

We proceed to the proof of Theorem 2. If  $1 < \alpha_1 < 3$ , then, without loss of generality, one may assume that  $f_\xi(\lambda) = c_1|\lambda|^{-\alpha_1}$  and  $f_\zeta(\lambda) = 0$  for  $|\lambda| \leq \lambda_0$ ,  $f_\zeta(\lambda) = c_2|\lambda|^{-\alpha_2}$  for  $|\lambda| > \lambda_0$ .

Further, using formula (6) and Szegő's theorem <sup>(13)</sup>, Theorem XIX) on the limiting distribution of the eigenvalues of a pair of Toeplitz forms, we directly obtain

$$\lim_{n \rightarrow \infty} J_n(\xi, \eta) = \infty,$$

which is equivalent to the assertion of orthogonality of the corresponding measures. In the case  $\alpha_1 = 3$ , Theorem 2 can be proved from its validity for  $\alpha_1 = 2$  by means of arguments related to those used in the proof of Theorem 1 in the case  $\alpha_1 \neq 2$ .

The orthogonality of the measures  $m_\xi$  and  $m_\eta$  can also be proved with the aid of theorems on the "almost sure" properties of sample functions of the considered processes and generalizing the known results of Baxter <sup>(5)</sup> and Gladyshev <sup>(7)</sup>. Below we shall give two new theorems of this type, in which, instead of Baxter's functional  $u(x_n) = x'_n x_n$ , there appears the more general quadratic functional  $Q_\chi(x_n) = x'_n A_n(\chi)x_n$ , where  $\chi(t)$  is a certain auxiliary process. In what follows, by  $x_n$  we shall denote the vector of increments of the process  $\eta(t)$  on the intervals  $\Delta t_j$  ( $j = 1, 2, \dots, n$ ).

**Theorem 3.** Suppose that, for large  $|\lambda|$ ,

$$f_\xi(\lambda) \leq \frac{c_1}{|\lambda|^{\alpha_1}}, \tag{7}$$

$$f_\zeta(\lambda) = \frac{c_2}{|\lambda|^{\alpha_2}} + o\left(\frac{1}{|\lambda|^{\alpha_2}}\right), \tag{8}$$

where  $2 \leq \alpha_1 + \frac{1}{2} = \alpha_2 \leq \frac{7}{2}$ . Then, if

$$f_X(\lambda) = \frac{1}{|\lambda|^{5-\alpha_2}} + o\left(\frac{1}{|\lambda|^{5-\alpha_2}}\right),$$

$n = 3^{m^2}$  ( $m = 1, 2, \dots$ ), then with probability 1

$$\lim_{m \rightarrow \infty} \frac{n^2}{4\pi \ln n} \left\{ Q_X(x_n) - \right. \\ \left. -16 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda_1}{2n} f_\xi(\lambda_1) d\lambda_1 \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\lambda_1 - \lambda_2}{2}}{\sin^2 \frac{\lambda_1 - \lambda_2}{2n}} \sin^2 \frac{\lambda_2}{2n} f_X(\lambda_2) d\lambda_2 \right\} = c_2. \quad (9)$$

**Theorem 4.** Suppose that, for large  $|\lambda|$ ,

$$f_\xi(\lambda) \leq \frac{c_1}{|\lambda|^{\alpha_1}}, \quad (10)$$

$$f_\zeta(\lambda) = \frac{c_2}{|\lambda|^{\alpha_2}} + o\left(\frac{1}{|\lambda|^{\alpha_2}}\right), \quad (11)$$

where  $\max\left[1, \alpha_2 - \frac{1}{2}\right] < \alpha_1 < \min[3, \alpha_2]$ . Then, if

$$f_X(\lambda) = \frac{1}{|\lambda|^\beta} + o\left(\frac{1}{|\lambda|^\beta}\right),$$

where  $1 < \beta \leq 3$ ,  $\alpha_2 + \beta < 5$ ,  $n = 3^m$  ( $m = 1, 2, \dots$ ), then with probability 1

$$\lim_{m \rightarrow \infty} \frac{n^{\alpha_2 + \beta - 3}}{K} \left\{ Q_X(x_n) - \right. \\ \left. -16 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda_1}{2n} f_\xi(\lambda_1) d\lambda_1 \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\lambda_1 - \lambda_2}{2}}{\sin^2 \frac{\lambda_1 - \lambda_2}{2n}} \sin^2 \frac{\lambda_2}{2n} f_X(\lambda_2) d\lambda_2 \right\} = c_2, \quad (12)$$

where

$$K = 32\pi \int_{-\pi}^{\pi} \sin^4 \frac{\mu}{2} \left[ \sum_{k=-\infty}^{\infty} \frac{1}{|\mu + 2k\pi|^{\alpha_2}} \right] \left[ \sum_{k=-\infty}^{\infty} \frac{1}{|\mu + 2k\pi|^{\beta}} \right] d\mu. \quad (13)$$

The method of proof of Theorems 3 and 4 is based on the application of the correlation theory of processes with stationary increments and of the apparatus of series.

Fourier (primarily Nikol'skii's theorem<sup>(14)</sup> on the order of the remainder of the Fourier sums of  $2\pi$ -periodic functions under various smoothness conditions).

For the proof of Theorem 3 the mathematical expectation of  $Q_\chi(x_n)$  is computed:

$$\begin{aligned} MQ_\chi(x_n) &= \text{Sp} [A_n(\xi) + A_n(\chi)] A_n(\chi) \\ &= 16 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda_1}{2n} f_\xi(\lambda_1) d\lambda_1 \int_{-\infty}^{\infty} \frac{\sin^2 \frac{\lambda_1 - \lambda_2}{2}}{\sin^2 \frac{\lambda_1 - \lambda_2}{2n}} \sin^2 \frac{\lambda_2}{2n} f_\chi(\lambda_2) d\lambda_2 + \frac{4\pi c_2 \ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right). \end{aligned} \quad (14)$$

Next, for the variance of  $Q_\chi(x_n)$  one can obtain the formula

$$\begin{aligned} DQ_\chi(x_n) &= 2 \text{Sp} [A_n(\eta) A_n(\chi)]^2 \\ &= 2^9 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda_1}{2n} f_\eta(\lambda_1) d\lambda_1 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda_2}{2n} f_\chi(\lambda_2) d\lambda_2 \int_{-\infty}^{\infty} \sin^2 \frac{\lambda_3}{2n} f_\eta(\lambda_3) d\lambda_3 \times \\ &\quad \times \int_{-\infty}^{\infty} \frac{\sin \frac{\lambda_1 - \lambda_2}{2} \sin \frac{\lambda_2 - \lambda_3}{2} \sin \frac{\lambda_3 - \lambda_4}{2} \sin \frac{\lambda_4 - \lambda_1}{2}}{\sin \frac{\lambda_1 - \lambda_2}{2n} \sin \frac{\lambda_2 - \lambda_3}{2n} \sin \frac{\lambda_3 - \lambda_4}{2n} \sin \frac{\lambda_4 - \lambda_1}{2n}} \sin^2 \frac{\lambda_4}{2n} f_\chi(\lambda_4) d\lambda_4. \end{aligned} \quad (15)$$

Using this expression, one can show that under the conditions of Theorem 3

$$DQ_\chi(x_n) \leq O\left(\frac{\ln n}{n^4}\right). \quad (16)$$

In view of (14) and (16), the assertion of Theorem 3 follows immediately from the Chebyshev and Borel-Cantelli lemmas.

Theorem 4 can be proved analogously.

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