Reports of the Academy of Sciences of the USSR

Yu. S. KOLESOV and M. A. KRASNOSEL' SKII

1962

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1962. Volume 145, No. 6

MATHEMATICS

Yu. S. KOLESOV and M. A. KRASNOSEL' SKII

LYAPUNOV STABILITY AND EQUATIONS WITH CONCAVE OPERATORS

(Presented by Academician I. G. Petrovskii on 28 III 1962)

Consider the system of differential equations

$$dx_i/dt = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1}$$

whose right-hand sides are periodic in t with period ω:

$$f_i(t+\omega, x_1, \dots, x_n) \equiv f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n). \tag{2}$$

Assume that for system (1) the uniqueness theorem for the solution holds for every initial condition and that every solution can be continued to the interval $0 \le t \le \omega$.

Write equation (1) in vector form

$$dx/dt = f(t, x) \tag{3}$$

and denote by $q(t, x_0)$ the solution of equation (3) satisfying the initial condition

$$q(0, x_0) = x_0. \tag{4}$$

It is clear that the fixed points of the Poincaré–Andronov transformation

$$Ux = q(\omega, x) \tag{5}$$

of the n-dimensional space E of points $x = \{x_1, \dots, x_n\}$ into itself are initial conditions for ω-periodic solutions of system (1).

Suppose that the operator (5) leaves invariant some cone K in the space E. Then, to prove the existence of periodic solutions of system (1), one can apply the general theory of nonlinear positive operators. Some of the theorems obtained in this way are given in ([1]). In a report at the IV All-Union Mathematical Congress, one of the authors noted that the concavity of the operator (5) can be used to investigate the stability of a periodic solution. The present note is devoted to the development of this idea.

All subsequent constructions are carried out under the assumption that for all t and nonnegative $x_1, \dots, x_n$

$$f_i(t, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \geq 0 \quad (i = 1, \dots, n). \tag{6}$$

1. Suppose that

$$f_i(t, x_1, \dots, x_n) \leq \sum_{k=1}^{n} b_{ik}(t) x_k + c_i \quad (x_1, \dots, x_n \geq 0;\ i = 1, \dots, n), \tag{7}$$

where c_i are nonnegative constants, and the functions $b_{ik}(t)$ are continuous and ω-periodic. We shall call inequality (7) the condition for the decrease of large solutions if the spectrum of the monodromy matrix (see ([2])) of the linear system

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} b_{ik}(t) y_k \quad (i = 1, \dots, n) \tag{8}$$

lies in the disk of radius $\rho < 1$.

Theorem 1. *If the conditions for the decrease of large solutions are satisfied, then system (1) has at least one nonnegative ω-periodic solution.*

Suppose, in addition, that

$$f_i(t, 0, \dots, 0) \equiv 0 \qquad (0 \leq t \leq \omega;\ i = 1, \dots, n). \tag{9}$$

Then system (1) has the trivial zero periodic solution. We shall be interested in the question of the existence of nonzero periodic solutions.

Assume that, for small nonnegative $x_1, \dots, x_n$, the inequalities

$$f_i(t, x_1, \dots, x_n) \geq \sum_{k=1}^{n} a_{ik}(t) x_k \qquad (i = 1, \dots, n), \tag{10}$$

hold, where the functions $a_{ik}(t)$ are continuous and ω-periodic, and the functions $a_{ik}(t)$ are nonnegative for $i \neq k$. Suppose there exists a finite sequence of numbers

$$i_1, i_2, \dots, i_m \qquad (m \geq n), \tag{11}$$

composed of all the numbers $1, 2, \dots, n$, such that

$$i_1 \neq i_2, \quad i_2 \neq i_3, \dots, \quad i_{m-1} \neq i_m, \quad i_m \neq i_1, \tag{12}$$

and the function

$$\Phi(t) = a_{i_1 i_2}(t) a_{i_2 i_3}(t) \cdots a_{i_{m-1} i_m}(t) a_{i_m i_1}(t) \tag{13}$$

does not vanish identically. If, moreover, the monodromy matrix of the linear system

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} a_{ik}(t) y_k \tag{14}$$

has an eigenvalue that is greater than 1, then inequalities (10) will be called the condition of growth of small solutions.

Theorem 2. *Let system (1) have a zero solution. Suppose the conditions of growth of small and decrease of large solutions are satisfied. Then system (1), besides the zero solution, has at least one ω-periodic solution whose components are nonnegative functions.*

2. Suppose that, for small nonnegative $x_1, \dots, x_n$, the inequalities

$$f_i(t, x_1, \dots, x_n) \leq \sum_{k=1}^{n} c_{ik}(t) x_k \qquad (i = 1, \dots, n), \tag{15}$$

hold, where $c_{ik}(t)$ are continuous and ω-periodic. We shall call these inequalities the condition of decrease of small solutions if the spectrum of the monodromy matrix of the linear system

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} c_{ik}(t) y_k \qquad (i = 1, \dots, n) \tag{16}$$

lies in the circle of radius $\rho < 1$. Suppose that, for all nonnegative $x_1, \dots, x_n$, the inequalities

$$f_i(t, x_1, \dots, x_n) \geq \sum_{k=1}^{n} e_{ik}(t) x_k - h_i \qquad (i = 1, \dots, n), \tag{17}$$

hold, where h_i are nonnegative constants, the functions $e_{ik}(t)$ are continuous and ω-periodic, and $e_{ik}(t)$ are nonnegative for $i \neq k$. Assume that there exists a sequence of numbers (11), composed of all the numbers $1, 2, \ldots, n$ and satisfying inequalities (12), and such that the function

$$\psi(t) = e_{i_1 i_2}(t) e_{i_2 i_3}(t) \cdots e_{i_{m-1} i_m}(t) e_{i_m i_1}(t) \tag{18}$$

does not vanish identically. We shall call inequalities (17) the condition

of growth of large solutions, if the monodromy matrix of the system

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} e_{ik}(t) y_k \qquad (i = 1, \ldots, n) \tag{19}$$

has an eigenvalue that is greater than 1.

Theorem 3. *Suppose that system (1) has the zero solution. Suppose that the conditions of decay of small solutions and growth of large solutions are satisfied. Then system (1), in addition to the zero solution, has at least one ω-periodic solution whose components are nonnegative functions.*

Theorem 3 covers classes of nonlinear systems containing substantial nonlinearities (for example, of exponential type). We emphasize that in the hypotheses of Theorem 3 there is no assumption that each solution of the system is extendable to some fixed interval of variation of t.

3. Suppose that the right-hand sides of system (1) are continuously differentiable with respect to $x_1, \ldots, x_n$. In the subsequent constructions an essential role is played by the matrix

$$I_0 = \left(\left(\frac{\partial}{\partial x_i} f_k(t, x_1, \ldots, x_n) \right) \right). \tag{20}$$

It is assumed that the off-diagonal elements of this matrix are nonnegative for positive $x_1, \ldots, x_n$.

Introduce the notation

$$F_i(t, x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k} f_i(t, x_1, \ldots, x_n) - f_i(t, x_1, \ldots, x_n). \tag{21}$$

Below it is assumed that these functions are nonpositive, and that one of them is negative for some $t = t^*$ and arbitrary positive $x_1, \ldots, x_n$.

Construct a finite number of matrices according to the following rule. The first of them is matrix (20). To obtain the next n matrices, in (20) the elements of one k-th row must be replaced by the function F_k. The subsequent matrices are obtained from (20) by replacing two rows by the functions F_k (in row number k) and F_l (in row number l). Analogously, matrices are constructed with three rows replaced, with four rows replaced, and so on.

Suppose that it is possible to indicate such a sequence (11), made up of all the numbers $1, 2, \dots, n$, that inequalities (12) are satisfied and the product of the elements $g_{i_1 i_2}(t, x_1, \dots, x_n)$, $g_{i_2 i_3}(t, x_1, \dots, x_n), \dots$, $g_{i_{m-1} i_m}(t, x_1, \dots, x_n)$, $g_{i_m i_1}(t, x_1, \dots, x_n)$ of one of the matrices listed above is different from zero for some $t = t_1$ and for positive $x_1, \dots, x_n$. If all the conditions listed in this paragraph are satisfied, then we shall say that the right-hand sides of system (1) possess the property of strong concavity.

We shall say that the right-hand sides of system (1) satisfy the condition of strong positivity if it is possible to indicate such a sequence (11), made up of all the numbers $1, 2, \dots, n$, that inequalities (12) are satisfied and the function

$$\begin{aligned}\chi(t, x_1, \dots, x_n) = {} & f_{i_1}(t, 0, \dots, 0, x_{i_2}, 0, \dots, 0)\times \\ & \times f_{i_2}(t, 0, \dots, 0, x_{i_3}, 0, \dots, 0) \cdots f_{i_{m-1}}(t, 0, \dots, 0, x_{i_m}, 0, \dots, 0)\times \\ & \times f_{i_m}(t, 0, \dots, 0, x_{i_1}, 0, \dots, 0)\end{aligned}$$

at $t = t_2$ and positive $x_1, \dots, x_n$ is different from zero.

Theorem 4. *If the right-hand sides of system (1) possess the property of strong concavity and satisfy the condition of strong positivity, then the operator (5) is u_0-concave on the cone K of vectors with nonnegative components, where $u_0 = \{1, 1, \dots, 1\}$.*

4. In what follows we shall say that the right-hand sides of system (1) are properly concave if the operator (5) is u_0-concave. Theorem 4, thus, gives sufficient conditions for proper concavity.

Theorem 5. Suppose the conditions of Theorem 1 are satisfied and the right-hand sides of system (1) are properly concave. Suppose system (1) has no zero solution. Then system (1) has a unique nonnegative ω-periodic solution, which is asymptotically stable in the sense of Lyapunov.

The components of the periodic solution in question in Theorem 5 are strictly positive functions.

Theorem 6. Suppose the conditions of Theorem 2 are satisfied and the right-hand sides of system (1) are properly concave. Then system (1) has a unique nonnegative and nonzero ω-periodic solution, which is asymptotically stable in the sense of Lyapunov. The components of this stable periodic solution are strictly positive.

Theorem 7. Suppose the conditions of Theorem 5 or Theorem 6 are satisfied. Then every solution $x(t)$ of system (1) satisfying a nonzero and nonnegative initial condition has the property

$$\lim_{t\to\infty} \|x(t) - x^*(t)\| = \lim_{t\to\infty} \sum_{i=1}^{n} |x_i(t) - x_i^*(t)| = 0, \tag{22}$$

where $x^*(t)$ is a positive periodic solution of system (1).

5. As an example, consider the system

$$dx/dt = -x^2(2 + \sin t) + \sqrt{y}, \qquad dy/dt = \sqrt{x} - y^2(1 + \sin^2 t). \tag{23}$$

It follows from Theorem 5 that it has a positive asymptotically stable periodic solution.

Sometimes, before applying the theorems of the present paper, it is convenient to make suitable changes of variables in the system. As a rule, this has to be done when studying equations of higher order. As an example, we state one sufficient condition for the existence of a stable periodic solution of the second-order equation

$$d^2x/dt^2 = f(t, x, dx/dt) \tag{24}$$

with right-hand side periodic in t. Suppose that $f(t, 0, 0) \equiv 0$, that for $x \geq 0,\ y \geq -x$,

$$f(t, x, y) \leq ax + by + c,$$

where $a < 0,\ b \leq a - 1$, and that for $0 \leq x,\ y + x \leq \rho_0$,

$$f(t, x, y) \geq \alpha x + \beta y,$$

where ρ_0 is some positive number, $\alpha > 0,\ \beta < a - 1$. Finally, suppose that for $x \geq 0,\ y \geq -x$ the inequalities

$$f'_x(t, x, y) - f'_y(t, x, y) \geq 1$$

and

$$xf'_x(t, x, y) + yf'_y(t, x, y) < f(t, x, y)$$

are satisfied. Then equation (24) can be rewritten as a system of two equations for which the conditions of Theorems 5, 6, 7 are satisfied, if one introduces the variables $u = x,\ v = x + dx/dt$. Thus, the conditions listed guarantee the existence of a nonzero stable periodic solution of system (24).

Analogous conditions can be written down for equations of higher order. It would be interesting to carry out an analogous study for systems that leave invariant (under shifts along trajectories) other cones (for example, circular ones). Theorems 5–7 used the theory of u_0-concave operators; analogous studies can be carried out by finding conditions under which operator (5) belongs to other classes of concave operators for which the theorems on the uniqueness of a nonzero fixed point and on the convergence of successive approximations to this fixed point are valid (see (1)).

Voronezh State University

Received
23 III 1962

REFERENCES CITED

1. M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, 1962.

2. F. R. Gantmakher, *The Theory of Matrices*, 1953.

Note: Figure translations are in progress. See original paper for figures.