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Abstract

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On the Impact of a Viscoplastic Rod against a Rigid Barrier

Nonstationary problems of the motion of viscoplastic media have been considered in the works of a number of investigators ([1–4]). A detailed analysis of the available exact and approximate solutions of nonstationary problems of viscoplastic flow is given in the monograph by A. Kh. Mirzadzhan-zade ([4]). In the present note, the formulation and an effective approximate solution are given for the problem of the impact against a rigid barrier of a viscoplastic rod of finite length.

1°. The problem is posed as follows. A rod of finite length l, made of a viscoplastic material and moving translationally in the direction of its axis (which we choose as the x-axis) with velocity $-v_0$, at the initial moment $t = 0$ strikes a rigid barrier $x = 0$.

We shall regard the motion of the rod as quasi-one-dimensional, i.e., we average the stresses, velocities, etc., over the cross-section of the rod. The relation between the stress σ and the strain rate $\partial v/\partial x$ is written for the case under consideration in the form

$$\frac{\partial v}{\partial x} = \frac{\sigma + \sigma_0}{\mu} \quad (|\sigma| > \sigma_0); \qquad \frac{\partial v}{\partial x} = 0 \quad (|\sigma| < \sigma_0), \tag{1}$$

where $v(x, t)$ is the velocity of the given cross-section of the rod at time t, $\sigma_0 > 0$ is the limiting stress, and μ is the coefficient of viscosity of the material; obviously, $\sigma \leqslant 0$ at all points of the rod.

For $t > 0$ the picture of the motion has the following form. The velocity of propagation of elastic disturbances in the medium under consideration is infinitely large; therefore the disturbance immediately encompasses the entire rod, and the velocity of motion for any $t > 0$ differs from $-v_0$ at all points of the rod. The rod is divided into two parts: in one part ($0 \leqslant x \leqslant x_0(t)$)—the viscoplastic region—the stresses in magnitude exceed σ_0 and viscoplastic flow takes place; in the other part ($x_0(t) < x \leqslant l$)—the rigid region—the stresses in magnitude are less than σ_0, so that this part of the rod moves as a rigid body. At the unknown moving boundary of the viscoplastic and rigid regions $x = x_0(t)$, the stresses and velocities are continuous.

2°. Introduce the dimensionless variables

$$\xi = \frac{x}{l}, \qquad \tau = \frac{\mu t}{\rho l^2}, \qquad u(\xi, \tau) = -\frac{v(x,t)}{v_0}, \qquad \xi_0(\tau) = \frac{x_0(t)}{l}. \tag{2}$$

In the viscoplastic region the function $u(\xi, \tau)$ satisfies the heat-conduction equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} \quad (0 \leqslant \xi \leqslant \xi_0(\tau)), \tag{3}$$

and in the rigid region it satisfies the equation

$$\frac{\partial u}{\partial \xi} = 0 \quad (\xi_0(\tau) \leqslant \xi \leqslant 1). \tag{4}$$

Integrating (4), we find

$$u(\xi, \tau) = u_0(\tau) \quad (\xi_0(\tau) \leqslant \xi \leqslant 1), \tag{5}$$

where $u_0(\tau) = v_0(t)/v_0$ is the dimensionless velocity of motion of the rigid region, as yet an undetermined function of time.

The equation of motion of the rigid region of the rod has the form:

$$M\frac{dv_0}{dt} = \rho F\,[l - x_0(t)]\frac{dv_0}{dt} = [\sigma_{x=x_0(t)-0}]F, \tag{6}$$

where M is the mass of the rigid part of the rod, and F is the cross-sectional area of the rod. Using the continuity condition for stress at the moving boundary $x = x_0(t)$ and passing to dimensionless variables, we reduce relation (6) to the form

$$\frac{du_0(\tau)}{d\tau} = -\frac{s}{1 - \xi_0(\tau)}, \tag{7}$$

where $s = \sigma_0 l/\mu v_0$ is the Saint-Venant parameter, a dimensionless combination of the defining parameters that characterizes the motion. By virtue of the continuity of stress and velocity at the moving boundary $x = x_0(t)$, we have

$$u[\xi_0(\tau), \tau] = u_0(\tau), \qquad \frac{\partial u[\xi_0(\tau), \tau]}{\partial \xi} = 0. \tag{8}$$

Fig. 1

There are also the obvious boundary and initial conditions:

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$\begin{aligned} u(0,\tau) &= 0 \quad (\tau > 0), \\ u(\xi,0) &= 1 \quad (0 < \xi \le 1), \\ u_0(0) &= 1, \qquad \xi_0(0) = 0. \end{aligned} \tag{9}$$

Fig. 2

Thus, the problem under consideration has been reduced to determining the functions $u(\xi,\tau)$, $u_0(\tau)$, and $\xi_0(\tau)$, satisfying relations (3), (5), (7), (8), and (9), i.e. to a problem with a moving boundary for the heat-conduction equation, which is not reducible to the traditional boundary-value problems of mathematical physics.

3°. For an approximate solution of the resulting system, we shall use the idea of the Kármán–Pohlhausen method from boundary-layer theory (5); namely, we represent the function $u(\xi,\tau)$ approximately in the form

$$u(\xi,\tau) = \begin{cases} 2u_0(\tau)\dfrac{\xi}{\xi_0(\tau)} - u_0(\tau)\dfrac{\xi^2}{\xi_0^2(\tau)} & (0 \le \xi \le \xi_0(\tau)), \\ u_0(\tau) & (\xi_0(\tau) < \xi \le 1). \end{cases} \tag{10}$$

If the functions $u_0(\tau)$ and $\xi_0(\tau)$ satisfy the last two conditions (9), then the function (10) satisfies all the conditions (9). Naturally, the function (10) in the viscoplastic region does not satisfy equation (3) exactly; we shall require that it satisfy this equation on average, i.e. that it satisfy the integral relation obtained by integrating (3) over the entire viscoplastic region $(0 \le \xi \le \xi_0(\tau))$:

$$\frac{d}{d\tau}\int_0^{\xi_0(\tau)} u(\xi,\tau)\,d\xi - u_0(\tau)\frac{d\xi_0}{d\tau} = -\left(\frac{\partial u}{\partial \xi}\right)_{\xi=0}. \tag{11}$$

Using (10) and (7), we obtain from this the differential equation

$$\frac{d\xi_0}{d\tau} = \frac{6}{\xi_0(\tau)} - \frac{2s\xi_0(\tau)}{[1-\xi_0(\tau)]u_0(\tau)}, \tag{12}$$

which, together with equation (7) and the last two conditions (9), determines the unknown functions $u_0(\tau)$ and $\xi_0(\tau)$, and thus the approximate

Fig. 3

Figure 3: Fig. 3

Fig. 4

Figure 4: Fig. 4

Fig. 3

solution of the problem. Introducing new dependent variables $p = u_0(\tau)/s,\ q = \xi_0^2(\tau)$, we rewrite the system of equations (7), (10) in the following form:

$$\frac{dq}{d\tau} = 12 - \frac{4q}{p(1-\sqrt{q})}, \qquad \frac{dp}{d\tau} = -\frac{1}{1-\sqrt{q}}, \tag{13}$$

which does not contain the Saint-Venant parameter s. The initial conditions will take, respectively, the form:

$$p(0) = \frac{1}{s}, \qquad q(0) = 0. \tag{14}$$

Fig. 4

Dividing the first equation (13) term by term by the second, we obtain a first-order equation not containing the independent variable τ:

$$\frac{dq}{dp} = -12(1-\sqrt{q}) + \frac{4q}{p}. \tag{15}$$

A simple qualitative investigation shows that in the region of interest to us ($p \geq 0,\ 0 \leq q \leq 1$) the integral curves behave as shown in Fig. 1. The problem under consideration corresponds to curves of class I, lying below the separatrix and having one maximum smaller than unity–only these curves intersect the abscissa axis at finite points and make it possible to satisfy condition (14). The corresponding integral curve intersects the abscissa axis at the point $p = 1/s$; the direction of motion of the representative point along the integral curve as time increases is shown in Fig. 1 by arrows.

4°. The investigation makes it possible to draw the following qualitative conclusions. At the beginning of the motion the viscoplastic region expands; its size $\xi_0(\tau)$ increases until it reaches a certain maximum, less than unity,

at $\tau = \tau_0(s)$ (Fig. 2), after which it begins to decrease. Thus, in all cases a certain part of the rod adjoining the free boundary remains undeformed. At some instant $\tau = \tau_1(s)$ the viscoplastic region disappears; this same instant corresponds to the velocity $u_0(\tau)$ of the rigid region of the rod becoming zero (Fig. 3), so that the motion of the rod ceases completely.

In the general case, system (13) requires numerical integration for its solution; the results of the integration for several values of the Saint-Venant parameter are presented in Figs. 2 and 3. For the case of large and small values of the Saint-Venant parameter, the solution can be represented in explicit form.

The approximate solution obtained makes it possible to determine the shape of the rod after impact. From the condition of incompressibility of the rod material we have

$$F = F_0 \left(1 + \frac{\partial U}{\partial x}\right)^{-1}, \tag{16}$$

where F is the cross-sectional area of the deformed rod corresponding to a certain value of x, U is the instantaneous longitudinal displacement, and F_0 is the cross-sectional area of the undeformed rod. At the instant when the impact ends, for an arbitrary section x we have:

$$\frac{\partial U}{\partial x} = -r \int_{\tau_*(\xi)}^{\tau_{**}(\xi)} \frac{\partial u(\xi,\tau)}{\partial \xi}\, d\tau = -2r \int_{\tau_*(\xi)}^{\tau_{**}(\xi)} \frac{u_0(\tau)\,[\xi_0(\tau) - \xi]}{\xi_0^2(\tau)}\, d\tau = -\frac{F - F_0}{F} = -2rf(\xi), \tag{17}$$

where $\tau_*(\xi)$ and $\tau_{**}(\xi)$ are the roots of the equation $\xi = \xi_0(\tau)$, $\tau_{**}(\xi) \geqslant \tau_*(\xi)$, $r = \rho v_0 l/\mu$ is the Reynolds parameter. In Fig. 4, for various values of the Saint-Venant parameter s, graphs are plotted of the function $f(\xi)$, which characterizes the change in the shape of the rod after impact. In an entirely analogous way, the shape of the rod can be determined at an arbitrary instant of the collision.

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Note: Figure translations are in progress. See original paper for figures.