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Abstract

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MATHEMATICS

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THE FOURIER METHOD FOR A HYPERBOLIC EQUATION WITH DISCONTINUOUS COEFFICIENTS

(Presented by Academician I. G. Petrovskii on 30 VI 1961)

In the present paper, by means of the Fourier method we prove the existence of a classical solution of a mixed problem for a hyperbolic equation with discontinuous coefficients. Here the surfaces on which the coefficients of the equations are discontinuous are subjected only to the Lyapunov condition, while the coefficients of the equations, the boundary surface, and the data of the problem are subjected to such smoothness requirements which, in the particular case when discontinuities of the coefficients are absent, pass into the conditions found in the work ⁽¹⁾, i.e. into the sharpest of the conditions known up to now for the solvability of the mixed problem for a hyperbolic equation with smooth coefficients.

1°. Let an N -dimensional open domain g , together with its boundary Γ , be contained in some open domain T . Suppose further that inside the domain g there lie k closed pairwise nonintersecting surfaces C_1, C_2, \dots, C_k , dividing the domain g into $k + 1$ domains g_0, g_1, \dots, g_k . (We denote by g_i ($i = 1, 2, \dots, k$) the domain lying inside the surface C_i .) Consider $k + 1$ self-adjoint elliptic operators

$$L_l u = \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left[a_{ij}^{(l)}(x) \frac{\partial u}{\partial x_j} \right] + c^{(l)}(x)u, \quad (1)$$

defined in the following domains: L_0 in $(T - g_1 - \dots - g_k)$, L_i ($i = 1, 2, \dots, k$) in $(g_i + C_i)$.

Let us denote by the symbol Ω the open cylinder $\Omega = g \times (0 < t < t_0]$, and by Ω_l ($l = 0, 1, \dots, k$) the open cylinder $\Omega_l = g_l \times [0 < t < t_0]$; by the symbols $\bar{\Omega}$ and $\bar{\Omega}_l$ the corresponding closed cylinders; by the symbol S_i ($i = 1, 2, \dots, k$) the cylindrical surface $S_i = C_i \times [0 \leq t \leq t_0]$; and by the symbol S_0 the cylindrical surface $S_0 = \Gamma \times [0 \leq t \leq t_0]$.

Consider in the cylinder Ω the mixed problem for a hyperbolic equation with discontinuous coefficients:

$$L_l u - u_{tt} = -f(x, t) \quad \text{in } \Omega_l \quad (l = 0, 1, \dots, k),$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad (2)$$

$$u|_{S_0} = 0, \quad [u]|_{S_i} = 0, \quad \left[\frac{\partial u}{\partial \nu} \right] \Big|_{S_i} = 0 \quad (i = 1, 2, \dots, k);$$

where

$$[u]|_{S_i} = u|_{S_i-0} - u|_{S_i+0}, \quad \left[\frac{\partial u}{\partial \nu} \right] \Big|_{S_i} = \frac{\partial u}{\partial \nu_i} \Big|_{S_i-0} - \frac{\partial u}{\partial \nu_0} \Big|_{S_i+0};$$

the symbols S_i-0 and S_i+0 mean that the limiting values are taken respectively from the inner and from the outer (with respect to the cylinder Ω_i) sides of the surface S_i ; ν_i denotes the conormal, inner with respect to the domain Ω_i , for the operator L_i ($i = 1, 2, \dots, k$); ν_0 denotes the conormal, outer with respect to Ω_0 , for the operator L_0 .

It is natural to call a **classical solution** of problem (2) a function $u(x, t)$ satisfying the following requirements: 1) $u(x, t)$ is continuous in the closed cylinder $\bar{\Omega}$; 2) all first and second derivatives ...

derivatives of the function $u(x, t)$ are continuous in each of the open cylinders Ω_l ($l = 0, 1, \dots, k$); 3) the first-order derivatives are continuous in $(\Omega_i + S_i)$ ($i = 1, 2, \dots, k$) and in $(\bar{\Omega}_0 - S_0)$; 4) the derivative $u_t(x, t)$ is continuous for $t = 0$, $x \in g$; 5) $u(x, t)$, in the classical sense, satisfies all the conditions of problem (1).

In [2] we established conditions A under which there exists a complete orthonormal system of classical eigenfunctions of the problem with discontinuous coefficients*

$$L_l v + \lambda v = 0 \quad (\text{in } g_l, \quad l = 0, 1, \dots, k),$$

$$v|_{\Gamma} = 0, \quad [v]|_{C_i} = 0, \quad \left[\frac{\partial v}{\partial \nu} \right] \Big|_{C_i} = 0 \quad (i = 1, 2, \dots, k). \quad (3)$$

In the same paper it is proved that, under conditions A, the complete orthonormal system of classical eigenfunctions of problem (3) coincides with the complete system of generalized eigenfunctions of this problem.

In complete analogy with what is set forth in Ch. 2 of [1], the following two theorems are proved.

Theorem 1. *If conditions A are fulfilled, then there can exist only one classical solution of the mixed problem (2).*

Theorem 2. *Let conditions A be fulfilled and, in addition, the following conditions: 1) $\varphi \in C^{(0)}$ in $(\bar{g} + \Gamma)$, $\varphi \in C^{(1,\mu)}$ in g_0 ; 2) $\psi \in C^{(0)}$ in $(\bar{g} + \Gamma)$; 3) $f \in C^{(0)}$ in $\bar{\Omega}$. Then, if there exists a classical solution of problem (2), this solution belongs to $W_2^1(\bar{\Omega})$ and coincides with the generalized, in the sense of O. A. Ladyzhenskaya ([3], pp. 72-73), solution of problem (2).*

It is known that the mixed problem (2) describes the vibrations of a bounded volume g with inhomogeneous filling. From this point of view, Theorem 2 proves the existence, for the vibrating particles, of finite energy for almost all moments of time t from the segment $0 \leq t \leq t_0$. We note that, for the proof of Theorem 2, in addition to the method of [1] and the results of [2], Theorem 2 of [4] is also used essentially.

2°. We pass to the proof of existence of a classical solution of the mixed problem (2). If the above conditions A are fulfilled, i.e., if there exist classical eigenfunctions of problem (3), then one may try to solve the mixed problem (2) by the Fourier method. Formal application of the Fourier method leads to the following series:

$$u(x, t) = \sum_{n=1}^{\infty} v_n(x) \left\{ \varphi_n \cos \sqrt{\lambda_n} t + \frac{\psi_n}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} t + \frac{1}{\sqrt{\lambda_n}} \int_0^t f_n(\tau) \sin \sqrt{\lambda_n} (t - \tau) d\tau \right\}. \quad (4)$$

Here λ_n are the eigenvalues, $v_n(x)$ the corresponding orthonormal eigenfunctions of problem (3), and φ_n , ψ_n , and $f_n(t)$ are the Fourier coefficients in the expansion of the functions $\varphi(x)$, $\psi(x)$, and $f(x, t)$ with respect to the system $v_n(x)$.

Definition. Let a certain function $F(x)$ ($F(x, t)$) belong to the class $W_2^{(2)}$ in each of the domains \bar{g}_l ($\bar{\Omega}_l$) ($l = 0, 1, \dots, k$). We shall say that this function satisfies the **conjugation conditions** on the surfaces C_i (S_i), if almost everywhere on C_i (S_i) there hold

* Conditions A are as follows: 1) the surface Γ is regular, the surfaces C belong to the Lyapunov class; 2) the coefficients of the operators L_l belong to the classes $a_{ij}^{(l)} \in C^{(1,\mu)}$, $C^{(l)} \in C^{(0,\mu)}$ in the domains of definition of the operators and, moreover, $C^{(l)} \leq 0$.

equalities

$$[F(x)]|_{C_i} = 0, \quad \left[\frac{\partial F}{\partial \nu} \right] \Big|_{C_i} = 0 \quad \left([F(x, t)]|_{S_i} = 0, \quad \left[\frac{\partial F}{\partial \nu} \right] \Big|_{S_i} = 0 \right)$$

$$(i = 1, 2, \dots, k).$$

Suppose that the following four conditions are satisfied:

- 1) The surface Γ is regular*, and the surfaces C_i ($i = 1, 2, \dots, k$) satisfy the Lyapunov conditions.
- 2) The coefficients of the operators L_l ($l = 0, 1, \dots, k$), first, satisfy the conditions A indicated above and, in addition, in the closed domain g_l ($l = 0, 1, \dots, k$) the coefficients $a_{ij}^{(l)}$ have continuous** derivatives up to order $[N/2] + 2$, and the coefficients $c^{(l)}$ up to order $[N/2] + 1$.
- 3) $\varphi \in W_2^{([N/2]+3)}$, $\psi \in W_2^{([N/2]+2)}$ in each of the domains g_l ($l = 0, 1, \dots, k$), and, moreover, $\varphi(x)$ and $\psi(x)$ are such that each of the functions*** $\varphi, L\varphi, \dots, L^{[(N+4)/4]}\varphi; \psi, L\psi, \dots, L^{[(N+2)/4]}\psi$ satisfies the conjugacy conditions on the surfaces C_i and belongs to $\overset{\circ}{D}(g)$.
- 4) $f \in W_2^{([N/2]+2)}$ in each of the cylinders $\bar{\Omega}_l$ ($l = 0, 1, \dots, k$) and, moreover, $f(x, t)$ is such that each of the functions $f, Lf, \dots, L^{[(N+2)/4]}f$ satisfies the conjugacy conditions on the surfaces S_i and belongs to $\overset{\circ}{D}_1(\Omega)$.

We shall call these four conditions **conditions B**.

Theorem 3. *If conditions B are satisfied, then there exists a (and moreover unique) classical solution of the mixed problem (2), defined by the series (4).*

In passing, together with the proof of Theorem 3 we give a justification of the Fourier method for problem (2). Namely, we establish that, under conditions B, the series (4) itself, as well as the series obtained by differentiating the series (4) once and twice with respect to t , converge uniformly in the closed cylinder $\bar{\Omega}$; the series obtained by differentiating (4) once with respect to any of the coordinates x_1, x_2, \dots, x_N converge uniformly in each of the cylinders $(\Omega_i + S_i)$ ($i = 1, 2, \dots, k$) and in the cylinder Ω'_0 , where Ω'_0 denotes the closed cylinder $\bar{\Omega}_0$ from which an arbitrarily small neighborhood of the surface S_0 has been removed; finally, the series obtained by termwise differentiating the series (4) twice with respect to any variables converge uniformly in any strictly interior subdomain of each of the cylinders Ω_l ($l = 0, 1, \dots, k$).

Let us briefly outline the proof of Theorem 3 and the justification of the Fourier method. Invoking the estimate of the Green function of the Dirichlet problem with discontinuous coefficients, obtained in work (2), and obtaining, by analogous means, estimates for the derivatives of the Green function, we, using the method of Ch. 3 of work (1), arrive at the following lemma.

Lemma 1. *Let conditions A be satisfied. Then the bilinear series of eigenfunctions of problem (3) of the form*

$$\sum_{n=1}^{\infty} \frac{v_n^2(x)}{\lambda_n^{[N/2]+1}}$$

converges uniformly in the closed

* The surface Γ is called **regular** if, in the domain bounded by this surface, the Dirichlet problem for the Laplace equation is solvable for any continuous boundary function.

** For a star-shaped domain g , the requirement of continuity of the derivatives of the coefficients $a_{ij}^{(l)}(x)$, and $c^{(l)}(x)$ may be replaced by the requirement $a_{ij}^{(l)} \in W_2^{([N/2]+2)}$, $c^{(l)} \in W_2^{([N/2]+1)}$ in each of the domains \bar{g}_l ($l = 0, 1, \dots, k$). We note that the latter requirement is less restrictive than that obtained by S. L. Sobolev (5), p. 222, for the solvability of the Cauchy problem with smooth coefficients, and than that obtained by O. A. Ladyzhenskaya (3), p. 101, for the solvability of the mixed problem with smooth coefficients.

*** Here by the operator L one should understand the operator with discontinuous coefficients, equal to L_l in the domain g_l ($l = 0, 1, \dots, k$).

of that domain $(g + \Gamma)$; the bilinear series of first derivatives of the eigenfunctions of the form

$$\sum_{n=1}^{\infty} \frac{[\partial v_n(x)/\partial x_i]^2}{\lambda_n^{[N/2]+2}}$$

converges uniformly in each of the domains $(g_i + C_i)$ ($i = 1, 2, \dots, k$) and in the domain g'_0 , where g'_0 denotes the closed domain \bar{g}_0 from which an arbitrarily small neighborhood of the surface Γ has been removed; finally, the bilinear series of second derivatives of the eigenfunctions of the form

$$\sum_{n=1}^{\infty} \frac{[\partial^2 v_n(x)/\partial x_i \partial x_j]}{\lambda_n^{[N/2]+3}}$$

converges uniformly in any strictly interior subdomain of each of the domains g_l ($l = 0, 1, \dots, k$).

In complete analogy with what was set forth in Chapter 4 of paper (1), one proves

Lemma 2. If conditions B are fulfilled, then one may assert the convergence of the numerical series

$$\sum_{n=1}^{\infty} \varphi_n^2 \lambda_n^{[N/2]+3}, \quad \sum_{n=1}^{\infty} \psi_n^2 \lambda_n^{[N/2]+2}, \quad \sum_{n=1}^{\infty} \lambda_n^{[N/2]+2} \int_0^t f_n^2(\tau) d\tau$$

and the uniform convergence in the closed cylinder of the series

$$\sum_{n=1}^{\infty} f_n(t) v_n(x).$$

After this, series (4) and the series obtained by differentiating (4) once and twice are investigated by elementary means analogous to those set forth in Chapter 5 of paper (1).

Remark 1. The existence of a classical solution of the mixed problem for a hyperbolic equation with discontinuous coefficients of a more general form than (2) was proved in the work of Yu. V. Egorov (6), but only under the assumption that the coefficients of the equation and the surfaces C_i of discontinuity of the coefficients satisfy smoothness conditions, extremely high in comparison with conditions B, which grow without bound as the number of dimensions increases.

Remark 2. The results obtained in the present work carry over to the case when part of the surfaces C_i lies inside others, and also to the case of boundary conditions of the second and third kind

$$(\partial u / \partial \nu_0 + hu)|_{s_0} = 0,$$

where $h(x) \geq 0$ (in this case it is additionally required that Γ satisfy the Lyapunov conditions).

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Note: Figure translations are in progress. See original paper for figures.

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