ON THE DIRICHLET PROBLEM FOR A CERTAIN CLASS OF ELLIPTIC SYSTEMS

A u_{xx} + 2B u_{xy} + C u_{yy} = 0

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Abstract

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ON THE DIRICHLET PROBLEM FOR A CERTAIN CLASS OF ELLIPTIC SYSTEMS

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In the paper ([2]) the Dirichlet problem for the unit disk with center at the origin was considered for a system of equations of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0 \tag{1}$$

in the case when the number of equations $n = 2$ and the characteristic polynomial $A + 2B\lambda + C\lambda^2 = 0$ has one pair of complex roots $(i, -i)$ of multiplicity n.

Under the same assumptions concerning system (1) and the vector-function $f(t)$, prescribed on the boundary of the disk, in the present note the case is considered when the number of equations n is any finite number. It is proved that the condition of weak connectedness of system (1), introduced by A. V. Bitsadze ([1]), is a necessary and sufficient condition for the Fredholm property of the Dirichlet problem. Moreover, it is possible to write the solution of the Dirichlet problem in explicit form:

$$u(x, y) = (z\bar{z} - 1)\,\mathrm{Re}\left[\sum_{k=1}^{n-1} P_k(z, \bar{z}) + \sum_{k=n-1}^{\infty} P_k(z, \bar{z})\right] + \mathrm{Re}\,\frac{1}{\pi i}\int_\Gamma \frac{f(t)}{t - z}\,dt - \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t}\,dt, \tag{2}$$

where $f(t)$ is a vector-function prescribed on the boundary of the disk; $z = x + iy$, with $|z| < 1$; $P_k(z, \bar{z})$ is a vector composed of polynomials with respect to the degrees of z and $\bar{z}$.

If k is even, then

$$P_{k-1}(z, \bar{z}) = {\sum_{i=0}^{k}}' \sum_{\substack{m+l=i \\ m \geq l}} \alpha_k^{\left[\frac{m-l}{2}\right]+1,\, m+l} M_k^{\left[\frac{m-l}{2}\right]+1} z^k \bar{z}^l \tag{3}$$

(the prime means that $i = 0, 2, \dots, k$).

If k is odd, then

$$P_{k-1}(z,\bar{z}) = \sum_{i=1}^{k}{}' \sum_{\substack{m+l=i \\ m\geq l}} \alpha_k^{\left[\frac{m-l}{2}\right]+1,\, m+1} M_k^{\left[\frac{m-l}{2}\right]+i_1} z^k \bar{z}^l \tag{3'}$$

(the prime means that $i = 1, 3, \ldots, k$).

Moreover, if $k \geq n-1$, then all $P_k(z,\bar{z})$ have coefficients at $\bar{z}^l z^k$ identically equal to zero as soon as $l \geq n-3$.

$$\alpha_k^{\left[\frac{m-l}{2}\right]+1,\, m+l}$$

are numbers which are found from the following linear algebraic system:

$$\alpha_{s+1}^{p,s} \frac{1 \cdot 2 \cdot \ldots \cdot ([s/2]+2-p)}{(s+2) \cdot \ldots \cdot ([s/2]+p+1)} = 1,$$

$$\begin{gathered} \frac{2 \cdot 3 \ldots ([s/2]+3-p)}{(s+3) \ldots ([s/2]+p+2)} \left[\alpha_{s+1}^{p,s} + \frac{1}{s+4}\alpha_{s+3}^{p,s}\right] = 1, \\ \frac{3 \cdot 4 \ldots ([s/2]+4-p)}{(s+4) \ldots ([s/2]+p+3)} \left[\alpha_{s+1}^{p,s} + \frac{2}{s+5}\left[\alpha_{s+3}^{p,s} + \frac{1}{s+6}\alpha_{s+5}^{p,s}\right]\right] = 1, \end{gathered} \tag{4}$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

where s, p are arbitrary integers. Since system (4) is triangular, all α are found uniquely. $M_k^{\left[\frac{m-i}{2}\right]+1}$ are constant vectors, which are found by solving the following algebraic system:

$$\begin{pmatrix} 2(A+C) & (A+2Bi-C) & 0 & 0 & 0 & \\ (A+2Bi-C) & 2(A+C) & (A-2Bi-C)\ldots & 0 & 0 & \\ 0 & (A+2Bi-C) & 2(A+B) & \ldots & 0 & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\ 0 & 0 & 0 & \ldots & 2(A+C) & (A-2Bi-C) \\ 0 & 0 & 0 & \ldots(A+2Bi-C) & 2(A+B) & \end{pmatrix} \tag{5}$$

where $(A+2Bi-C)$, $2(A+C)$, $(A-2Bi-C)$ are matrices of order n. Consequently, the total order of the coefficient matrix of system (5) is nk, where k is the number of matrix rows. If k is odd, then the vector of unknowns has the form

$$\left(M_{k-1}^{(k-1)/2} \ldots M_{k-1}^{2},\ \operatorname{Re} M_{k-1}^{1},\ \overline{M}_{k-1}^{2} \ldots \overline{M}_{k-1}^{(k-1)/2}\right).$$

Thus, to each $P_k(z,\bar z)$ there corresponds its own system (5). It has been shown that, if system (1) is weakly coupled, then for all $k \geq n-1$ the determinant of system (5) is always $\neq 0$. It follows that, knowing $f(t)$, one can uniquely construct the polynomials $P_{n-1}(z,\bar z)$, $P_n(z,\bar z)$, ..., and only for a finite number of polynomials $P_1(z,\bar z)$, $P_2(z,\bar z), \dots, P_{n-2}(z,\bar z)$ can the determinant of system (5) vanish.

Thus, the question of the solvability of the Dirichlet problem has been reduced to the question of the solvability of a finite number of linear algebraic systems, and, consequently, the weak coupling of system (1) ensures the Fredholm property of the Dirichlet problem.

The number of linearly independent nontrivial solutions of the homogeneous Dirichlet problem is equal to

$$\left[\frac{(n-2)n(n-1)}{2} - \sum_{k=1}^{n-2} r_k\right], \tag{6}$$

where r_k is the rank of the determinant of system (5).

For solvability of the nonhomogeneous Dirichlet problem, the same number of additional conditions is imposed on $f(t)$:

$$\operatorname{Re}\left((A+2Bi-C)\frac{1}{\pi i}\int_\Gamma \frac{f(t)}{t^{k+1}}\,dt,\ M_{k-1}^{k/2'}\right) = 0, \qquad k = 1, 2, \dots, n-2, \tag{7}$$

where $M_{k-1}^{k/2'}, \dots, M_{k-1}^{1'}$ are solutions of the adjoint homogeneous system; their number is equal to $(nk - r_k)$; the outer parentheses denote the scalar product of two vectors.

If system (1) is strongly coupled, then the determinant of system (5) is always equal to zero for all $k \geq n-1$. Consequently, the nonhomogeneous Dirichlet problem is unsolvable, while the homogeneous Dirichlet problem has infinitely many solutions.

It can be shown that a necessary and sufficient condition for weak coupling of system (1) is the condition

$$\begin{vmatrix} 2(A+C) & (A-2Bi-C) & 0 & \cdots & 0 \\ (A+2Bi-C) & 2(A+C) & (A-2Bi-C) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2(A+C) \end{vmatrix} \neq 0, \tag{8}$$

where the number of matrix rows is $k = n-1$.

Below we give an example showing that, under weak coupling of system (1), the homogeneous Dirichlet problem has nontrivial solutions.

Indeed, let

$$A = \left\| \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix} \right\|, \quad C = \left\| \begin{matrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{matrix} \right\|, \quad B = \left\| \begin{matrix} 0 & 0 & -3/\sqrt{14} \\ -3/\sqrt{14} & -3/\sqrt{14} & -10/\sqrt{14} \\ \sqrt{7/2} & \sqrt{7/2} & \sqrt{7/2} \end{matrix} \right\|.$$

The condition of weak coupling in this case is:

$$\left| \begin{matrix} 2(A+C) & (A-2Bi-C) \\ (A+2Bi-C) & 2(A+C) \end{matrix} \right| \neq 0.$$

It is easy to verify that the system is weakly coupled. Nevertheless, the homogeneous Dirichlet problem has a nontrivial solution:

$$u_0(x,y) = (x^2+y^2-1)M,$$

where M is a solution of the homogeneous system $(A+C)M=0$. Since $\det|A+C|=0$, one can find $M \neq 0$.

In conclusion I express my deep gratitude to my advisor, Corresponding Member of the Academy of Sciences of the USSR A. V. Bitsadze, for posing the problem and for assistance in carrying out this work.

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1. A. V. Bitsadze, *Equations of Mixed Type*, 1959, p. 64.

2. E. V. Zolotareva, DAN, **132**, No. 4, 751 (1960).

Note: Figure translations are in progress. See original paper for figures.