CORRESPONDENCE OF BOUNDARIES UNDER Q-QUASICONFORMAL MAPPINGS OF A BALL

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CORRESPONDENCE OF BOUNDARIES UNDER Q-QUASICONFORMAL MAPPINGS OF A BALL

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In the present note we investigate the boundary correspondence arising under homeomorphic Q-quasiconformal ([1]) mappings of a ball R in N-dimensional ($N \geq 2$) Euclidean space E^N.

Definition 1. A sequence $\{U_n\}$ of subdomains of a domain D is called **regular** if:
a) $U_{n+1} \subset U_n,\ n = 1, 2, \dots;$
b) $\left[\bigcap_{n=1}^{\infty} \overline{U_n}\right] \subset \Gamma$, where Γ is the boundary of D;
c) the part S_n of the boundary Γ_n of any domain $U_n \in \{U_n\}$, consisting of points of the domain D (the relative boundary of U_n in D), is a connected set;
d) the distance $\rho_D(S_n, S_{n+1})$ in the domain ([2]) between the relative boundaries of any two neighboring domains of the sequence is positive;
e) there exists no more than one attainable (see, for example, ([2])) boundary point of the domain D which is an attainable boundary point for each of the domains of the sequence $\{U_n\}$.

Definition 2. Two sequences of domains $\{U'_n\}$, $\{U''_n\}$ will be called **equivalent** if every member of either of the sequences under consideration contains all members of the other sequence with sufficiently large indices.

Lemma 1. *If $\{U_n\}$ is a sequence of subdomains of the ball R, satisfying conditions a) and d) of regularity, and $\{p_m\}$ is a sequence of points of R converging to some point of the continuum $K = \bigcap_{n=1}^{\infty} \overline{U_n}$, then any domain $U_n \in \{U_n\}$ contains all points $p_m \in \{p_m\}$, beginning with some index $m(n)$.*

If the lemma were false, then there would exist a sequence of points $\tilde{p}_m \in R$, not belonging to some domain $U_n \in \{U_n\}$, and converging to some point $p \in K$. But the point p is a limit point for interior points of the domain U_n; therefore it would be a limit point for the relative boundary S_n of the domain U_n in R. For the same reason the point p would be a limit point for the relative boundary S_{n+1} of the domain U_{n+1} in R, and we would obtain

$$\rho_R(S_n, S_{n+1}) = 0,$$

which contradicts the condition. Lemma 1 is proved.

Corollary 1. If $\{U_n\}$ is a sequence of subdomains of the ball R satisfying conditions a), b), and d) of regularity, then all points of the continuum

$$K = \bigcap_{n=1}^{\infty} \overline{U_n}$$

are attainable boundary points for each of the domains of the sequence.

Indeed, it follows from the lemma that on any curve $p = \lambda(t)$ $(0 \leq t < 1)$ going in the ball R to a point $p \in K$, one can indicate a point $p_n = \lambda(t_n)$ such that the curve $p = \lambda(t)$ $(t_n \leq t < 1)$ no longer goes outside the corresponding domain $U_n \in \{U_n\}$. Thus any point $p \in K$ is indeed an attainable boundary point for each of the domains of the sequence.

Hence, obviously, Corollaries 2 and 3 follow.

Corollary 2. A regular sequence of subdomains of the ball contracts to a point, i.e. the intersection $K = \bigcap\limits_{n=1}^{\infty} \overline{U}_n$ consists of a single point.

Corollary 3. Two regular sequences of subdomains of the ball are equivalent if and only if they contract to one and the same point of the boundary sphere.

Lemma 2. *Under a homeomorphic Q-quasiconformal mapping $p^* = T(p)$ of the ball R onto the domain D^*, a regular sequence $\{U_n\}$ of subdomains of R is carried into a regular sequence $\{U_n^*\}$ of subdomains of D^*; the preimage of a regular sequence of subdomains of D^* is a regular sequence of subdomains of R; equivalent sequences of subdomains are carried into equivalent ones.*

Let $\{U_n^*\}$ be the image of a regular sequence $\{U_n\}$ of subdomains of the ball R. To establish the regularity of the sequence $\{U_n^*\}$, it is enough to verify the fulfillment of conditions d) and e) of regularity, since conditions a), b), and c) are fulfilled by virtue of the homeomorphic nature of the mapping.

Suppose that for some two domains U_m^*, U_{m+1}^* of the sequence $\{U_n^*\}$ condition d) is not fulfilled. Then in D^* one can choose a sequence of arcs γ_k^* joining $S_m^* = T(S_m)$ and $S_{m+1}^* = T(S_{m+1})$ and contracting to some (possibly infinitely remote) point $p^* \in \Gamma^*$. The preimages of the arcs γ_k^* will be arcs $\gamma_k \subset R$ joining S_m and S_{m+1}. By the condition $\rho_R(S_m, S_{m+1}) > 0$, from the sequence of arcs γ_k one can choose a subsequence of arcs whose endpoints converge to two distinct points. The images of the arcs of this subsequence contract to a point, which contradicts Lemma 4 of ([2]). Thus $\rho_{D^*}(S_m^*, S_{m+1}^*) > 0$, and the fulfillment of condition d) has been verified.

If condition e) were not fulfilled, then the continuum $K = \bigcap\limits_{n=1}^{\infty} \overline{U}_n$, determined by the regular sequence $\{U_n\}$, by the theorem from ([2]) would contain at least two distinct points, which contradicts the second corollary of Lemma 1. The regularity of the sequence $\{U_n^*\}$ is verified.

Let now $\{U_n\}$ be a sequence of subdomains of the ball R which is the preimage of some regular sequence $\{U_n^*\}$ of subdomains of D^*. The proof of the regularity of the sequence $\{U_n\}$, obviously, again consists in verifying the fulfillment of conditions d) and e) of regularity.

Suppose that for some two terms U_m, U_{m+1} of the sequence $\{U_n\}$ condition d) is not fulfilled, i.e. $\rho_R(S_m, S_{m+1}) = 0$. Then, by Lemma 3 of ([2]), the quantity $\rho_{D^*}(S_m^*, S_{m+1}^*)$ would also be equal to zero, which is impossible. The fulfillment of condition d) is verified.

Finally, if the continuum $K = \bigcap_{n=1}^{\infty} \overline{U}_n$ were different from a point, then by the theorem from ([2]) it would have to contain at least two distinct points p_i $(i = 1, 2)$, corresponding to two different attainable boundary points (p_i^*, l_i^*) $(i = 1, 2)$ of the domain $\overline{D}^*$. The curves $l_i = T^{-1}(l_i^*)$ $(i = 1, 2)$ tend respectively to the points p_i $(i = 1, 2)$. But if the sequence of domains $U_n \subset R$ satisfies conditions a), b), c), and d) of regularity, then, according to Corollary 1 of Lemma 1, on each of the curves $\{p_i = l_i(t)\ (0 \leq t < 1)\}$ $(i = 1, 2)$ one can indicate such a point $p_{in} = l_i(t_{in})$ that the whole curve $\{p_i = l_i(t)\ (t_{in} \leq t < 1)\}$ will lie in the corresponding domain $U_n \in \{U_n\}$. The mapping $p^* = T(p)$ is homeomorphic, and therefore the curves $\{p_i = l_i(t)\ (t_{in} \leq t < 1)\}$ $(i = 1, 2)$ cannot leave the domain $U_n^* = T(U_n)$. This means that the different attainable points

(p_i^*, l_i^*) $(i = 1, 2)$ of the domain D^* are attainable boundary points for each of the domains of the regular sequence $\{U_n^*\}$. The resulting contradiction establishes the regularity of the sequence $\{U_n\}$.

The last assertion of the lemma on the passage from equivalent sequences to equivalent ones follows at once from the homeomorphy of the mapping.

Definition 3. The aggregate $(K, \{U_n\})$ of a continuum

$$K = \bigcap_{n=1}^{\infty} \overline{U}_n$$

and a regular sequence $\{U_n\}$ of subdomains of the domain D will be called a **boundary element** of the domain D. Two boundary elements $(K', \{U_n'\})$ and $(K'', \{U_n''\})$ are regarded as coincident if and only if the regular sequences $\{U_n'\}$ and $\{U_n''\}$ defining them are equivalent.

Thus, any of the equivalent regular sequences completely determines a certain boundary element of the domain.

From Lemmas 1 and 2 it follows:

Theorem 1. *For any homeomorphic Q-quasiconformal mapping $p^* = T(p)$ of the ball R onto the domain D^*, a one-to-one correspondence can be established between points of the boundary sphere Γ and boundary elements of the domain*

D^*, under which the boundary element $(K^*, \{U_n^*\})$ of the domain D^* corresponds to the point on Γ determined by the regular sequence

$$\{U_n\} = T^{-1}(\{U_n^*\}).$$

If additional conditions are imposed on the boundary of the domain D^*, then in accordance with these conditions we obtain the following two theorems.

Theorem 2. *In order that a homeomorphic Q-quasiconformal mapping of the ball R onto the domain D^* extend continuously to the closed ball $\overline{R}$, it is necessary and sufficient that every regular sequence of subdomains of D^* determine on the boundary D^* only one point.*

The necessity of the formulated condition is obvious. Let us prove its sufficiency.

Let p be some point of the sphere Γ bounding R, and let $(p^*, \{U_n^*\})$ be the boundary element of the domain D^* corresponding by Theorem 1 to the point p. By Lemma 1 the images of all sequences of points $p_n \in R$ converging to p have one and the same limit* p^*. Put $p^* = \overline{T}(p)$. To prove the continuity of this mapping in $\overline{R}$, it now suffices to prove its continuity on Γ. Let $\{p_n\}$ be an arbitrary sequence of points $p_n \in \Gamma$ converging to the point $p \in \Gamma$. The continuity of the mapping $\overline{T}(p)$ under approach to the boundary through points of the open ball has already been established; therefore, for each point $p_n \in \{p_n\}$ there is a point $\tilde{p} \in R$ such that $|\tilde{p}_n - p_n| < \frac{1}{n}$ and

$$\left|\overline{T}(\tilde{p}_n) - \overline{T}(p_n)\right| < \frac{1}{n}. \tag{1}$$

The sequence $\{\tilde{p}_n\}$ consists of interior points of R and, obviously, converges to the same point $p \in \Gamma$ as the sequence $\{p_n\}$; therefore, for any $\varepsilon > 0$ one can indicate a number $n(\varepsilon)$ such that for all $n > n(\varepsilon)$

$$\left|\overline{T}(\tilde{p}_n) - \overline{T}(p)\right| < \varepsilon/2. \tag{2}$$

Comparing (1) and (2), for $n > \max[n(\varepsilon), 2/\varepsilon]$, we obtain:

$$\left|\overline{T}(p_n) - \overline{T}(p)\right| < \varepsilon,$$

and the continuity of the mapping $p^* = \overline{T}(p)$ in $\overline{R}$ is proved.

Theorem 3. *In order that a Q-quasiconformal homeomorphism between the ball R and some domain D^* can be extended to a homeo–*

* Without loss of generality this limit may be regarded as finite.

of a homeomorphic mapping of the closed domains $\overline{R}, \overline{D}^*$, it is necessary and sufficient that each regular sequence of subdomains D^* determine on the boundary D^* only one point, and that inequivalent regular sequences converge to different points of the boundary D^*.

The necessity of these conditions is clear. Let us prove their sufficiency. From Theorem 2 it follows that the extended mapping $p^* = \overline{T}(p)$ is continuous in $\overline{R}$, and from Theorem 1 its one-to-one character in $\overline{R}$ follows. But a continuous one-to-one mapping of a compactum is homeomorphic, and Theorem 3 is proved.

In conclusion we give an example of a Q-quasiconformal mapping of a ball for which the domain corresponding to the ball has a boundary element $(K^*, \{U_n^*\})$ with continuum K^* different from a point.

It is easy to construct a Q-quasiconformal mapping of a ball $R \subset E^3$ onto the half-cylinder

$$D : \left[\left(x - \frac{e^{2\pi}+1}{2} \right)^2 + y^2 < \left(\frac{e^{2\pi}+1}{2} \right)^2,\ z > 0 \right],$$

where x, y, z are Cartesian coordinates. Next, by the Q-quasiconformal mapping

$$u = \left(1 + \frac{x}{1+z}\right) \cos \ln(1+z), \quad v = \frac{y}{1+z}, \quad w = \left(1 + \frac{x}{1+z}\right) \sin \ln(1+z)$$

$(u, v, w$ are Cartesian coordinates), we transform D into a domain D^*, which is a half-cylinder narrowing and twisted into a spiral, winding infinitely many times around the circle $C^* :\ [u^2 + w^2 = 1,\ v = 0]$. The boundary D^* contains points belonging to both one and two distinct boundary elements. Moreover, for one of the boundary elements $(K^*, \{U_n^*\})$ of the domain D^*, the continuum K^* coincides with the entire circle C^*.

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CITED LITERATURE

1. B. V. Shabat, DAN, **130**, No. 6 (1960).

2. V. A. Zorich, DAN, **145**, No. 1 (1962).

Note: Figure translations are in progress. See original paper for figures.