I. O. KHACHATRYAN

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Abstract

Full Text

I. O. KHACHATRYAN

ON WEIGHTED APPROXIMATION OF ENTIRE FUNCTIONS OF ZERO DEGREE BY POLYNOMIALS ON THE REAL AXIS

(Presented by Academician S. N. Bernstein on 12 III 1962)

Let $\varphi(t) \geq 1,\ -\infty < t < \infty$, be an arbitrary function such that

$$t^n \varphi^{-1}(t) \to 0 \quad \text{as } |t| \to \infty.$$

Consider the space C_φ^0 of continuous functions $f(t)$ on the axis $(-\infty, \infty)$ satisfying the condition

$$f(t)\varphi^{-1}(t) \to 0 \quad \text{as } |t| \to \infty.$$

The norm of an element is defined to be the number

$$\sup_{-\infty<t<\infty} |f(t)\varphi^{-1}(t)|.$$

It is obvious that polynomials belong to C_φ^0.

The problem posed by S. N. Bernstein in 1924 is the following: to find necessary and sufficient conditions on the function $\varphi(t)$ under which the polynomials form an everywhere dense set in the space C_φ^0. In this case the function $\varphi(t)$ is called a weight function. A number of works have been devoted to this problem and to its various generalizations. In particular, we mention the papers ([1−7]). A detailed exposition of these questions can be found in the survey articles ([2,4]).

Denote by $\mathfrak{M}$ the set of polynomials $P(t)$ for which

$$\|P(t)(t-i)^{-1}\| \leq 1,$$

and by $\psi(z)$:

$$\psi(z) = \sup_{P\in\mathfrak{M}} |P(z)|.$$

Then each of the following conditions is a necessary and sufficient condition for completeness:

$$\text{a)} \quad \sup_{P\in\mathfrak{M}} \int_{-\infty}^{\infty} \frac{\ln|P'(t)|}{1+t^2}\, dt = \infty; \qquad \text{b)} \quad \psi(z) \equiv \infty, \quad \operatorname{Im} z \neq 0;$$

$$\text{c)} \quad \int_{-\infty}^{\infty} \frac{\ln\psi(t)}{1+t^2}\, dt = \infty.$$

Condition a) was indicated by N. I. Akhiezer and S. N. Bernstein ([3]), and conditions b) and c) by S. N. Mergelyan ([4]).

Suppose that $\varphi(t)$ is not a weight function. Then, as was shown by S. N. Mergelyan ([4])*, polynomials can form a dense set only in the class $C_\varphi^* \subset C_\varphi^0$ of functions coinciding on E_φ with entire functions of zero degree ($t \in E_\varphi,\ \varphi(t) \neq \infty$).

In the present note a necessary and sufficient condition is given for the density of polynomials in C_φ^*.

Denote by $\mathfrak{M}_1$ the set of entire functions of zero degree satisfying the conditions

$$\|(t-i)^{-1}f(t)\| \le 1, \qquad f(t)(t-i)^{-1}\varphi^{-1}(t) \to 0 \quad \text{as } |t| \to \infty,$$

and by $\psi_1(z)$:

$$\psi_1(z) = \sup_{f \in \mathfrak{M}1} |f(z)|.$$

* S. N. Bernstein, as early as 1924 ([9]), for the special case of a weight, noted the alternative: either polynomials form a dense set in C_φ, or only entire functions of not too large growth are approximated by polynomials.

Theorem 1. Suppose that the polynomials do not form a dense set in C_φ^0. Then, in order that the polynomials form a dense set in C_φ^*, it is necessary and sufficient that

$$\psi_1(z) \equiv \psi(z), \qquad \operatorname{Im} z \neq 0. \tag{1}$$

Necessity. By the hypothesis of the theorem, the function $\psi(z)$ is uniformly bounded in every finite domain ([4]), and, consequently, the function $\theta(t) = |t-i|^{-1}\psi(t)$ is bounded on every finite interval.

Introduce a new norm

$$\|f\|_1 = \sup_{-\infty<t<\infty} \left|\theta^{-1}(t)\right| |f(t)|.$$

From the obvious inequality $\theta(t) \leqslant \varphi(t)$ we have

$$\|f\| \leqslant \|f\|_1.$$

Moreover, from the definition of $\theta(t)$, for $P \in \mathfrak{M}$ we shall have

$$\sup_{-\infty<t<\infty} \left|P(t)(t-i)^{-1}\theta^{-1}(t)\right| \leqslant 1, \qquad \text{i.e.} \quad \left\|P(t)(t-i)^{-1}\right\|_1 \leqslant 1.$$

The inequality $\|P(t)(t-i)^{-1}\| \leqslant C$ implies the inequality
$\|P(t)(t-i)^{-1}\|_1 \leqslant C$, i.e.

$$\left\| P(t)(t-i)^{-1} \right\| \geqslant \left\| P(t)(t-i)^{-1} \right\|_1 ,$$

and, consequently, for polynomials $P(t) = (t-i)Q(t)$ we obtain

$$\|Q(t)\| = \|Q(t)\|_1.$$

Using the density of the set of polynomials from $\mathfrak{M}$ in $\mathfrak{M}_1$, one can show that $\|f\| = \|f\|_1$ for every $f \in C_\varphi^*$.

For given z_0, $\operatorname{Im} z_0 \neq 0$, and $\delta > 0$, choose $f_\delta(z) \in C_\varphi^*$ so that

$$|f_\delta(z_0)| > \psi_1(z_0) - \delta.$$

Choose a polynomial $P_{\delta,n}(t) \in \mathfrak{M}$ so that

$$\|P_{\delta,n}(t) - f_\delta(t)\|_1 < \varepsilon,$$

where $\varepsilon > 0$ is a preassigned number. Then

$$|f_\delta(t) - P_{\delta,n}(t)| < a\varepsilon \qquad \left(t \in [0,1],\ a = \sup_{0 \leqslant t \leqslant 1} \theta(t) \right),$$

i.e.

$$\lim_{n\to\infty} P_{\delta,n}(t) = f_\delta(t) \qquad (0 \leqslant t \leqslant 1),$$

and from the convergence of the sequence $P_{\delta,n}(t)$ to $f_\delta(t)$ on $[0,1]$ there follows the uniform convergence of $P_{\delta,n}(z)$ to $f_\delta(z)$ in every bounded domain. Hence we have

$$\lim_{n\to\infty} |P_{\delta,n}(z_0)| = |f_\delta(z_0)|.$$

We have obtained that

$$\sup_{P\in\mathfrak{M}} |P(z_0)| \geqslant |f_\delta(z_0)| > \psi_1(z_0) - \delta,$$

and, consequently,

$$\psi(z_0) \geqslant \psi_1(z_0).$$

The reverse inequality is obvious, i.e. equality (1) holds.

Sufficiency. We shall prove that, under equality (1), a functional $\mathcal{F}$ in $C_\varphi^{*\prime}$ that vanishes on polynomials is identically zero. Let

$$\mathcal{F}[t^n] = 0, \qquad n = 0, 1, 2, \ldots .$$

Then, for any polynomial $P(t)$,

$$\mathcal{F}\left[\frac{P(t) - P(z)}{t - z}\right] = 0. \tag{2}$$

Extend the functional $\mathcal{F}$ to the space C_φ^0 of all continuous functions; then from (2) it follows that

$$\mathcal{F}\left[P(t)(t-z)^{-1}\right] = P(z)\mathcal{F}\left[(t-z)^{-1}\right]. \tag{3}$$

The function $\mathcal{F}(z) = \mathcal{F}[(t-z)^{-1}]$ is holomorphic for $\operatorname{Im} z \neq 0$. Next, using the general form of a linear functional in C_φ^0, we obtain the inequality

$$\left|\mathcal{F}\left[P(t)(t-z)^{-1}\right]\right| \leq \int_{+\infty}^{\infty} \left|P(t)(t-z)^{-1}\varphi^{-1}(t)\right| |d\sigma(t)| \quad \left(\operatorname{Var}_{-\infty<t<\infty} \sigma(t) < \infty\right).$$

For a given $\varepsilon > 0$, choose N so large that

$$\int_{|t|\geq N} |d\sigma(t)| < \varepsilon.$$

Then for any polynomial $P(t) \in \mathfrak{M}$,

$$\begin{aligned}
\left|\mathcal{F}\left[P(t)(t-z)^{-1}\right]\right| \leq \int_{-N}^{N} &\left|P(t)(t-z)^{-1}\varphi^{-1}(t)\right| |d\sigma(t)| + \\
&+ \int_{|t|>N} \left|P(t)(t-z)^{-1}\varphi(t)\right| |d\sigma(t)| \leq \\
&\leq \max_{|t|\leq N} \left|P(t)(t-i)^{-1}\varphi(t)(t-i)(t-z)^{-1}\right| |\operatorname{Var}\sigma(t)| + \varepsilon \left\|P(t)(t-z)^{-1}\right\| \leq \\
&\leq \max \left|(t-i)(t-z)^{-1}\right| + \varepsilon \left\|P(t)(t-z)^{-1}\right\|.
\end{aligned}$$

As $\operatorname{Im} z \to \infty$, the first term on the right-hand side of the last inequality tends to zero, and we obtain, as $\operatorname{Im} z \to \infty$,

$$\left|\mathcal{F}\left[P(t)(t-z)^{-1}\right]\right| = o(1)$$

for any polynomial $P(t) \in \mathfrak{M}$. Hence, and from (3), it follows that $|\mathcal{F}(z)P(z)| = o(1)$ for any $P(t) \in \mathfrak{M}$ as $\operatorname{Im} z \to \infty$. Therefore

$$|\mathcal{F}(z)| \leq o(1)\left[\sup_{P\in\mathfrak{M}} |P(z)|\right]^{-1} = o(1)\psi^{-1}(z). \tag{4}$$

Let $f(t)$ be an entire function of zero degree such that

$$\|f(t)(t-i)^{-1}\| \leq 1; \qquad f(t)(t-i)^{-1}\varphi^{-1}(t) \to 0 \quad \text{as } |t| \to \infty. \tag{5}$$

Then the quotient

$$\frac{f(t)-f(z)}{t-z} \in C^*_\varphi$$

for every z. It can be shown that the function

$$\Phi(z) = \mathcal{F}\left[\frac{f(t)-f(z)}{t-z}\right] = \mathcal{F}\left[f(t)(t-z)^{-1}\right] - f(z)\mathcal{F}(z) \tag{6}$$

is entire of zero degree. From condition (5) it follows that

$$\mathcal{F}\left[f(t)(t-iy)^{-1}\right] = o(1).$$

Then from (6) and (4) it follows that

$$|\Phi(iy)| \leq o(1) + |f(iy)|\,|\mathcal{F}(iy)| \leq o(1) + o(1)\psi^{-1}(iy) \sup_{P\in\mathfrak{M}} |f(iy)|.$$

The upper bound in the last inequality is taken in the class of entire functions of zero degree satisfying condition (5). Consequently,

$$|\Phi(iy)| \leq o(1) + o(1)\psi_1(iy)\psi^{-1}(iy) \quad \text{as } |y| \to \infty. \tag{7}$$

Since, by assumption, $\psi(z) \equiv \psi_1(z)$ ($\operatorname{Im} z \neq 0$), it follows from (7) that

$$|\Phi(iy)| = o(1) \quad \text{as } |y| \to \infty. \tag{8}$$

From (8) it follows that $\Phi(z) \equiv 0$, i.e.,

$$\mathfrak{F}\left[\frac{f(t)-f(z)}{t-z}\right] = 0$$

for all entire functions satisfying condition (5). In particular,

$$\mathfrak{F}[f(t)(t-z_0)^{-1}] = 0,$$

where z_0 is a zero of the function $f(z)$.

Let now $f(t)$ be from C^*_φ. Take the function

$$F(t) = \|f\|^{-1}(t-i)f(t).$$

It satisfies condition (5), and $F(i) = 0$; consequently,

$$0 = F[\mathfrak{F}(t)(t-i)^{-1}] = \mathfrak{F}[\|f\|^{-1}f(t)] = \|f\|^{-1}\mathfrak{F}[f(t)],$$

i.e., the functional $\mathfrak{F}$ is equal to zero on all entire functions of zero degree $f(t) \in C^*_\varphi$, and therefore also on all functions of the space C^*_φ.

From the main theorem we obtain:

Theorem 2. If $\varphi(t)$ is an entire even function with nonnegative coefficients:

$$\varphi(t) = \sum_0^\infty a_k t^{2k}, \qquad a_0 \geq 1, \quad a_k \geq 0, \quad k = 1, 2, \dots,$$

then polynomials are everywhere dense in C^*_φ.

Theorem 3. If the function $h(t)$ admits the representation

$$h(t) = h(0)\exp\left\{\int_0^{|t|} \omega(t)t^{-1}\,dt\right\},$$

in which $0 \leq \omega(t) < \infty$, $\omega(t)$ increases monotonically, then for any given $\varepsilon > 0$ and given function $f(t) \in C^*_h$ there exists a polynomial $P_f(t)$ such that

$$\frac{|f(t) - P_f(t)|}{(t^2+1)h(t)} < \varepsilon.$$

Let us note that one can construct examples of weights $\varphi(t)$ for which $\psi(z) \not\equiv \psi_1(z)$. In particular, such a weight will be

$$\varphi(t) = \begin{cases} |t-i|^{-1}\exp t^{\rho_1}, & \text{for } t \geq 0, \quad 0 < \rho_1 < \frac{1}{2}, \\ |t-i|^{-1}\exp t^{\rho_2}, & \text{for } t \leq 0, \quad \frac{1}{2} \leq \rho_2 < 1. \end{cases}$$

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Institute of Mathematics and Mechanics
Academy of Sciences of the Armenian SSR

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Note: Figure translations are in progress. See original paper for figures.