# Reports of the Academy of Sciences of the USSR

1962

SovietRxiv

**Abstract**

**Full Text**


## Reports of the Academy of Sciences of the USSR

1962, Volume 145, No. 6


### MATHEMATICS

**I. I. Bavrin**

# On the Uniqueness of Extremal Functions in Estimates of Taylor Coefficients of Bounded Functions of Two Complex Variables

*(Presented by Academician M. A. Lavrent' ev, 29 III 1962)*

In the case of two complex variables I have given exact estimates of Taylor coefficients for bounded functions ([1]). However, in these estimates the question of the uniqueness of the extremal functions was not clarified; this question is considered in the present note.

Let the function

$$F(w,z)=\sum_{m,n=0}^{\infty} a_{mn} w^m z^n \tag{1}$$

satisfy the following conditions: a) it is regular in the bicylinder $E\{|w|<R_1,\ |z|<R_2\}$; b) for almost every fixed $t_0$ from the segment $[0,2\pi]$, for $0<\rho<1$ the integral * is bounded

$$\int_0^{2\pi} \left|\psi(\rho e^{i\varphi},t_0)\right| d\varphi, \tag{2}$$

where $\psi(\rho e^{i\varphi},t)\equiv F(R_1\rho e^{i\varphi},R_2\rho e^{i(t-\varphi)})$. We shall denote this class of functions by $\lambda$ **.

As is known, a function $F(w,z)$, regular in $E$, in addition to its representation by the series (1), can be represented in $E$ by the diagonal series

$$F(w,z)=\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k} a_{k-l,l} w^{k-l} z^l\right). \tag{3}$$

It follows from conditions a) and b) that for almost all $t$ in $[0, 2\pi]$ the function $\psi(\zeta, t) \in H_1$, and therefore for almost all $t$ in $[0, 2\pi]$ it has, almost everywhere on $|\zeta| = 1$, definite limiting values along nontangential paths, which form the boundary function $\psi(e^{i\varphi}, t)$, and for almost all $t$ in $[0, 2\pi]$, according to Cauchy' s formula ***

$$\sum_{l=0}^{m+n} a_{m+n-l,l} R_1^{m+n-l} R_2^l e^{-ilt} = (2\pi)^{-1} \int_0^{2\pi} \psi(e^{i\varphi}, t) e^{-i(m+n)\varphi} \, d\varphi, \tag{4}$$

since, by virtue of the representation (3),

$$[(m+n)!]^{-1} \psi_\zeta^{(m+n)}(0, t) = \sum_{l=0}^{m+n} a_{m+n-l,l} R_1^{m+n-l} R_2^l e^{-ilt}. \tag{5}$$

Multiplying both sides of equality (4) by $e^{int}$, from the resulting equality, integrating with respect to $t$ from 0 to $2\pi$, we find

$$a_{mn} = (4\pi^2 R_1^m R_2^n)^{-1} \int_0^{2\pi} dt \int_0^{2\pi} \psi(e^{i\varphi}, t) e^{-i[(m+n)\varphi - nt]} \, d\varphi, \tag{6}$$

where the integrals are understood in the Lebesgue sense.

* That is, the set of those points $t_0$ in $[0, 2\pi]$ at which the boundedness of the integral (2) fails for $0 < \rho < 1$ has measure zero.
** Obviously, the class $q$ considered in ([2]) is contained in $\lambda$.
*** Here the integral is understood in the Lebesgue sense.

Since for almost all $t$ in $[0, 2\pi]$ we have $\psi(\zeta, t) \in H_1$, it follows that for almost all $t$ in $[0, 2\pi]$, $\psi(\zeta, t)\zeta^{m+n-1} e^{-int} \in H_1$ $(m + n = 1, 2, ...)$, and hence, by Riesz' theorem ([3]), for almost all $t$ in $[0, 2\pi]$,

$$\int_0^{2\pi} \psi(e^{i\varphi}, t) e^{i[(m+n-1)\varphi - nt]} \, de^{i\varphi} = 0,$$

where the integral is understood in the Lebesgue sense; whence

$$\int_0^{2\pi} dt \int_0^{2\pi} \psi(e^{i\varphi}, t) e^{i[(m+n-1)\varphi - nt]} \, de^{i\varphi} = 0, \tag{7}$$

where the integrals are understood in the Lebesgue sense. Thus we have proved:

**Theorem 1.** For functions

$$F(w, z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n \in \lambda$$

the formulas (6), (7) hold.

We shall say that a certain property $L$ holds for almost all points $(t,\varphi)$ of the square $S\{0 \le t \le 2\pi,\ 0 \le \varphi \le 2\pi\}$, or, briefly, almost everywhere on $S$, if almost all $t$ with $0 \le t \le 2\pi$ have the property that for them almost all $\varphi$ with $0 \le \varphi \le 2\pi$ have property $L$.

**Corollary 1.** If a function $F(w,z) \in \lambda$ is such that $\psi(e^{i\varphi},t) = 0$ almost everywhere on $S$, then $F(w,z) = 0$ in $E$.

**Remark 1.** Let the function $F(w,z)$ be regular in $E$. Then the function $\psi(\rho\zeta,t)$ $(0<\rho<1)$, for any fixed $t$, $0 \le t \le 2\pi$, is regular in $|\zeta| \le 1$, and therefore, by Cauchy' s formula,

$$[(m+n)!]^{-1}\rho^{m+n}\psi^{(m+n)}_{\zeta^{m+n}}(0,t) = (2\pi)^{-1}\int_0^{2\pi}\psi(\rho e^{i\varphi},t)e^{-i(m+n)\varphi}\,d\varphi.$$

Taking (5) into account, we conclude that the expression $[(m+n)!]^{-1}\left|\psi^{(m+n)}_{\zeta^{m+n}}(0,t)\right|$ is bounded as a function of $t$, continuous on the interval $[0,2\pi]$; this implies that the modulus of the integral

$$\int_0^{2\pi}\psi(\rho e^{i\varphi},t)e^{-i[(m+n)\varphi-nt]}\,d\varphi, \qquad 0 \le t \le 2\pi,$$

is bounded for $0<\rho<1$. This property of functions regular in $E$ was noted without proof in the note ([2]).

**Theorem 2.** If in the bicylinder $E$ the function

$$F(w,z) = \sum_{m,n=0}^{\infty} a_{mn}w^m z^n,$$

where $a_{00}$ is prescribed, is regular and $|F(w,z)| \le 1$, then for $m+n>0$

$$|a_{mn}| \le (1-|a_{00}|^2)R_1^{-m}R_2^{-n}, \tag{8}$$

and equality holds only for the function

$$F(w,z) = \left(a_{00}R_1^m R_2^n + \eta_{mn}w^m z^n\right)\left(R_1^m R_2^n + \bar{a}_{00}\eta_{mn}w^m z^n\right)^{-1}, \qquad |\eta_{mn}| = 1. \tag{9}$$

**Proof.** By the hypothesis of Theorem 2, in the disk $|\zeta|<1$, for any fixed $t$ with $0 \le t \le 2\pi$, the function $\psi(\zeta,t)$ is regular and $|\psi(\zeta,t)| \le 1$. Therefore the function $\psi(\zeta,t)$, for any fixed $t$ with $0 \le t \le 2\pi$, has almost everywhere on $|\zeta| = 1$ definite limiting values along nontangential paths, forming the boundary function $\psi(e^{i\varphi},t)$; moreover, for any $t$ with $0 \le t \le 2\pi$, almost everywhere on $|\zeta| = 1$,

$$|\psi(e^{i\varphi},t)| \le 1. \tag{10}$$

Obviously $F(w,z) \in \lambda$, and therefore, by Theorem 1, formulas (6), (7) hold. Using these formulas, for $m+n>0$ we have *

$$a_{mn}R_1^m R_2^n + 2a_{00}\bar{a}_{00}\eta_{mn} = (4\pi^2)^{-1}\int_0^{2\pi} dt \int_0^{2\pi} \psi(e^{i\varphi},t)e^{-i[(m+n)\varphi - nt]}K(\varphi,t)\,d\varphi,$$

where

$$K(\varphi,t) = \left(1 + \bar{a}_{00}e^{i[(m+n)\varphi - nt]}\eta_{mn}\right)^2.$$

* In the proof of Theorem 2 below, the author follows the method of G. M. Goluzin ([4]).

Hence, for $\eta_{mn} = e^{i\arg a_{mn}}$ (if $a_{00}=0$, then $\eta_{mn}$ is arbitrary with $|\eta_{mn}|=1$),

$$\begin{aligned} |a_{mn}|R_1^m R_2^n + 2|a_{00}|^2 &= (4\pi^2)^{-1}\left|\int_0^{2\pi} dt \int_0^{2\pi} \psi(e^{i\varphi},t)e^{-i[(m+n)\varphi - nt]}K(\varphi,t)\,d\varphi\right| \\ &\le (4\pi^2)^{-1}\int_0^{2\pi} \left|\int_0^{2\pi} \psi(e^{i\varphi},t)e^{-i[(m+n)\varphi - nt]}K(\varphi,t)\,d\varphi\right| dt \\ &\le (4\pi^2)^{-1}\int_0^{2\pi} dt \int_0^{2\pi} |\psi(e^{i\varphi},t)|\,|K(\varphi,t)|\,d\varphi \\ &\le (4\pi^2)^{-1}\int_0^{2\pi} dt \int_0^{2\pi} |K(\varphi,t)|\,d\varphi = 1 + |a_{00}|^2, \end{aligned} \tag{11}$$

which leads to estimate (8). Using the proposition for one complex variable* and the fact that equality in

$$\int_0^{2\pi} dt \int_0^{2\pi} |\psi(e^{i\varphi},t)|\,|K(\varphi,t)|\,d\varphi \le \int_0^{2\pi} dt \int_0^{2\pi} |K(\varphi,t)|\,d\varphi \tag{12}$$

holds only when $|\psi(e^{i\varphi},t)| = 1$ almost everywhere on $S$,** we conclude that the equality signs in (11) occur only in the case when

$$\arg\psi(e^{i\varphi},t)e^{-i[(m+n)\varphi-nt]}K(\varphi,t)=\text{const}\pmod{2\pi}$$

and $|\psi(e^{i\varphi},t)|=1$ almost everywhere on $S$. But these conditions can be written otherwise:

$$\arg\psi(e^{i\varphi},t)R(\varphi,t)=\text{const}\pmod{2\pi},\qquad|\psi(e^{i\varphi},t)R(\varphi,t)|=1,$$

where

$$R(\varphi,t)=\frac{R_1^mR_2^n+\bar a_{00}\eta_{mn}(R_1e^{i\varphi})^m(R_2e^{i(\varphi-t)})^n}{a_{00}R_1^mR_2^n+\eta_{mn}(R_1e^{i\varphi})^m(R_2e^{i(\varphi-t)})^n},$$

which is equivalent to the single condition

$$\psi(e^{i\varphi},t)R(\varphi,t)=\text{const}=e^{i\alpha}$$

almost everywhere on $S$, which, by virtue of Corollary 1 and the fact that $F(0,0)=a_{00}$, is fulfilled only in the case of the function (9), for which equality holds in (8).

**Theorem 3.** If in the domain $D^{***}$ the function

$$F(w,z)=\sum_{m,n=0}^{\infty}a_{mn}w^mz^n,$$

* The equality sign in

$$\left|\int_0^{2\pi}f(e^{i\varphi})\,d\varphi\right|\le\int_0^{2\pi}|f(e^{i\varphi})|\,d\varphi,$$

where $f(e^{i\varphi})$ and $|f(e^{i\varphi})|$ are summable on $[0,2\pi]$, is attained if and only if

$$\arg f(e^{i\varphi})=\text{const}\pmod{2\pi}$$

for almost all $\varphi$, $0\le\varphi\le 2\pi$. Indeed, let

$$\theta=\arg\int_0^{2\pi}f(e^{i\varphi})\,d\varphi.$$

Then

$$\left|\int_0^{2\pi}f(e^{i\varphi})\,d\varphi\right|=e^{-i\theta}\int_0^{2\pi}f(e^{i\varphi})\,d\varphi=\int_0^{2\pi}|f(e^{i\varphi})|\cos[\arg f(e^{i\varphi})-\theta]\,d\varphi\le\int_0^{2\pi}|f(e^{i\varphi})|\,d\varphi,$$

from which the assertion easily follows.

** In fact, bearing in mind (10), equality in (12), as is known, is obtained only when

$$\int_0^{2\pi} |\psi(e^{i\varphi}, t)|\, |K(\varphi, t)|\, d\varphi = \int_0^{2\pi} |K(\varphi, t)|\, d\varphi$$

for almost all $t \in [0, 2\pi]$. The latter equality, taking (10) into account, holds for each of the indicated almost all $t$ only when

$$|\psi(e^{i\varphi}, t)| = 1$$

for almost all $\varphi \in [0, 2\pi]$. Thus, equality in (12) holds only when

$$|\psi(e^{i\varphi}, t)| = 1$$

almost everywhere on $S$.

*** $D$ is a bounded complete bicircular domain (containing its center $(0,0)$), whose boundary is twice continuously differentiable and analytically convex from the outside. As A. A. Temlyakov proved (5), the boundary of this domain can be parametrically given in the form

$$|w| = r_1(\tau), \quad |z| = r_2(\tau), \quad 0 \le \tau \le 1,$$

where $r_1(0) = 0$, $r_1(1) < \infty$, $r_1'(\tau) > 0$ on $(0, 1]$, and

$$r_2(\tau) = \exp\left[-\int \frac{\tau}{1-\tau}\, d\ln r_1(\tau)\right] \quad (r_2(1) = 0).$$

where $a_{00}$ is given, is regular, and $|F(w, z)| \leqslant 1$, then for $m + n > 0$

$$|a_{mn}| \leqslant (1 - |a_{00}|^2) M^{-1}(m, n), \tag{13}$$

where

$$M(m, n) \equiv r_1^m \left(\frac{m}{m+n}\right) r_2^n \left(\frac{m}{m+n}\right)$$

(taking $0^0 = 1$); equality for $m > 0,\ n > 0$ occurs only for the function

$$F(w, z) = (a_{00} M(m, n) + \eta_{mn} w^m z^n)\big(M(m, n) + \overline{a}_{00} \eta_{mn} w^m z^n\big)^{-1}, \quad |\eta_{mn}| = 1. \tag{14}$$

**Proof.** Since the hypersurface $|w| = r_1(\tau)$, $|z| = r_2(\tau)$, $0 \leqslant \tau \leqslant 1$, is composed of the surfaces $|w| = r_1(\tau)$, $|z| = r_2(\tau)$ under a continuous change of the parameter $\tau$ in the segment $[0, 1]$, and since the point $(0, 0)$ is an interior point of the bicylinder $|w| < r_1(\tau)$, $|z| < r_2(\tau)$ for any $\tau$, $0 < \tau < 1$, it follows, by Theorem 2, that

$$|a_{mn}| \leqslant (1 - |a_{00}|^2) r_1^{-m}(\tau) r_2^{-n}(\tau),$$

$m > 0,\ n > 0$, where $\tau$ is an arbitrary number in $(0, 1)$. Using the functions of one complex variable

$$F(0, z) = \sum_{n=0}^{\infty} a_{0n} z^n,$$

$$F(w, 0) = \sum_{m=0}^{\infty} a_{m0} w^m,$$

we have

$$|a_{0n}| \leqslant (1 - |a_{00}|^2) r_2^{-n}(0), \quad n > 0; \qquad |a_{m0}| \leqslant (1 - |a_{00}|^2) r_1^{-m}(1), \quad m > 0.$$

It is not difficult to see that

$$\max_{0 \leqslant \tau \leqslant 1} r_1^m(\tau) r_2^n(\tau) = \begin{cases} r_1^m \left( \dfrac{m}{m+n} \right) r_2^n \left( \dfrac{m}{m+n} \right), & m > 0, \quad n > 0; \\ r_1^m(1), & m > 0, \quad n = 0; \\ r_2^n(0), & m = 0, \quad n > 0; \end{cases}$$

therefore we have the estimate (13). Further, in the case of the bicylinder

$$\left\{ |w| < r_1 \left( \frac{m_0}{m_0 + n_0} \right), \quad |z| < r_2 \left( \frac{m_0}{m_0 + n_0} \right) \right\}$$

$(m_0, n_0$ are any fixed ones from $m = 1, 2, \dots;\ n = 1, 2, \dots)$, the equality sign in the estimate

$$|a_{m_0, n_0}| \leqslant (1 - |a_{00}|^2) M^{-1}(m_0, n_0)$$

according to Theorem 2 occurs only for the function

$$F(w, z) = (a_{00} M(m_0, n_0) + \eta_{m_0, n_0} w^{m_0} z^{n_0}) \big( M(m_0, n_0) + \overline{a}_{00} \eta_{m_0, n_0} w^{m_0} z^{n_0} \big)^{-1},$$

$$|\eta_{m_0, n_0}| = 1,$$

and since $m_0, n_0$ are any fixed numbers from $m = 1, 2, \ldots;\ n = 1, 2, \ldots$, equality in (13) for $m > 0,\ n > 0$ occurs only for the function (14).

**Remark 2.** From Theorems 2 and 3, as corollaries, Cauchy' s inequalities for Taylor coefficients and the uniqueness of extremal functions follow easily.

**Remark 3.** In the case of $n$ complex variables all arguments are carried out in exactly the same way (for a polycylinder—as for a bicylinder; for the special class of domains (6)—as for the domain $D$).

Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

Received
22 III 1962

## REFERENCES

[1] I. I. Bavrin, DAN, **126**, No. 5 (1959).
[2] I. I. Bavrin, DAN, **137**, No. 3 (1961).
[3] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Moscow-Leningrad, 1952.
[4] G. M. Goluzin, *Matem. sborn.*, **26** (68), 7 (1950).
[5] A. A. Temlyakov, DAN, **120**, No. 5 (1958).
[6] I. I. Bavrin, DAN, **131**, No. 6 (1960).

*Note: Figure translations are in progress. See original paper for figures.*