ON THE SIGN OF THE GREEN’ S FUNCTION OF A LINEAR SECOND-ORDER DIFFERENCE BOUNDARY-VALUE PROBLEM

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ON THE SIGN OF THE GREEN' S FUNCTION OF A LINEAR SECOND-ORDER DIFFERENCE BOUNDARY-VALUE PROBLEM

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In the present note we consider the question of the sign of the Green' s function of the linear difference boundary-value problem

$$L[y] \equiv y(x+2) + P_1(x)y(x+1) + P_0(x)y(x) = \varphi(x), \qquad y(0) = y(r) = 0. \quad (1)$$

Here r is an integer; $P_1(x)$, $P_0(x)$, and $\varphi(x)$ are defined on the set $x = 0, 1, \dots, r-1$; $P_0(x) \neq 0$ at every point of this set.

By the Green' s function of problem (1) we mean the function $G(x, s)$, defined on the set $x = 0, 1, \dots, r$; $s = 0, 1, \dots, r-2$, which for every fixed s satisfies the boundary-value problem

$$L[G] = \delta_{x,s} \quad (\delta_{x,s} = 0 \text{ for } x \neq s,\ \delta_{s,s} = 1), \qquad G(0, s) = G(r, s) = 0. \quad (2)$$

The question under consideration is closely connected with theorems on difference inequalities of the type of Chaplygin' s theorem on differential inequalities (^{1}).

As is known (^{2,3}), the Green' s function of the boundary-value problem

$$y'' + q_1(x)y' + q_0(x)y = f(x), \qquad y(0) = y(r) = 0$$

in the square $0 < x < r$, $0 < s < r$ preserves its sign if and only if every nontrivial solution of the equation

$$y'' + q_1(x)y' + q_0(x)y = 0$$

has no more than one zero in the interval $[0, r]$; moreover, under the indicated conditions the Green' s function is negative. For the boundary-value problem (1) the analogous assertion does not hold; namely, under certain assumptions the Green's function of this problem on the set $Q : x = 1, 2, \dots, r-1;\ s = 0, 1, \dots, r-2$ may preserve its sign even when some solution of the equation $L[y] = 0$ has, on the set $x = 0, 1, \dots, r$, more than one zero (or one change of sign). In the latter case the Green' s function can only be positive.

Below we give necessary and sufficient conditions for preservation of the sign of the Green' s function of problem (1) on the set Q.

We introduce the following definitions.

Definition 1. The function $K(x, s)$, for any fixed s, satisfying the equation $L[K] = 0$ and the conditions $K(s, s) = 0$, $K(s + 1, s) = 1$, will be called the **Cauchy function** of the operator $L[y]$ (cf. (^{1})).

Definition 2. We shall call a point $x_0 \in g$ (g is the set of points $x = 0, 1, \dots, r$) a **critical point** of the function $f(x)$, defined on the set g, if $f(x_0) = 0$ or $f(x_0)f(x_0 + 1) < 0$.

The proof of Theorems 1 and 2 below rests essentially on the following lemma, which refines a theorem of M. A. Skalkina (^{4}):

Lemma. Let $x = n$, $x = m$ $(m \geq n + 2)$ be two consecutive zeros of a solution $y_1(x)$ of the equation $L[y] = 0$, and let $y_1(x)$ preserve its sign on the set $x = n + 1, n + 2, \dots, m - 1$.

If $P_0(x) > 0$ $(x = n, n + 1, \dots, m - 2)$, then every solution $y_2(x)$ of the equation $L[y] = 0$, linearly independent of $y_1(x)$, has on the set $x = n, n + 1, \dots, m$ one and only one critical point.

It should be noted that if the condition $P_0(x) > C$ $(x = n,\ n + 1, \dots, m - 2)$ is not satisfied, the lemma is false; namely, there exists a solution of the equation $L[y] = 0$, linearly independent of $y_1(x)$, which has at least two changes of sign on the set $x = n,\ n + 1, \dots, m$. And even under the condition $P_0(x) > 0$ $(x = n,\ n + 1, \dots, m - 2)$, any solution of the equation $L[y] = 0$ linearly independent of $y_1(x)$ will have at least two critical points on the set $x = n,\ n + 1, \dots, m$, provided only that $y_1(x)$ changes sign on the set $x = n + 1,\ n + 2, \dots, m - 1$. For example, one of the solutions $y_1(x)$ of the equation

$$y(x + 2) + P_1(x)y(x + 1) + P_0(x)y(x) = 0$$

$$(P_1(0) = P_1(1) = P_1(2) = P_1(3) = -1,\ P_1(4) = -2,$$

$$P_0(0) = P_0(1) = 2,\ P_0(2) = P_0(3) = P_0(4) = 1)$$

takes the values: $y_1(0) = 0$, $y_1(1) = y_1(2) = 1$, $y_1(3) = -1$, $y_1(4) = -2$, $y_1(5) = -1$, $y_1(6) = 0$. The solution $y_2(x)$, satisfying the initial conditions

$y_2(0) = 1,\ y_2(1) = 0$, is linearly independent of $y_1(x)$ and takes the values: $y_2(0) = 1,\ y_2(1) = 0,\ y_2(2) = y_2(3) = -2,\ y_2(4) = 0,\ y_2(5) = 2,\ y_2(6) = 4$, i.e. $y_2(x)$ has two zeros between two consecutive zeros of $y_1(x)$.

Thus, a zero-separation theorem, analogous to Sturm' s theorem, generally does not hold for difference equations.

Theorem 1. *The Green' s function of problem (1) exists and is nonpositive on the set Q if and only if*

$$P_0(x) > 0 \quad (x = 1, 2, \dots, r-2), \qquad K(x, 0) > 0 \quad (x = 1, 2, \dots, r).$$

Theorem 2. *The Green' s function of problem (1) exists and is nonnegative on the set Q if and only if*

$$P_0(x) > 0 \quad (x = 1, 2, \dots, r-2), \qquad K(x, 0) > 0 \quad (x = 1, 2, \dots, r-2),$$

$$K(r-1, 0) \geq 0, \qquad K(r, 0) < 0, \qquad K(x, 1) > 0 \quad (x = 2, 3, \dots, r-1),$$

$$K(r, 1) \geq 0.$$

Conditions for the positivity of the Cauchy function of a linear difference operation are considered in the works ([5],[6]).

In conclusion, let us consider the behavior of the Green' s function of problem (1) under $P_0(x) > 0$ $(x = 1, 2, \dots, r-2)$ in the case where the conditions of Theorem 1 or Theorem 2 are not fulfilled.

Theorem 3. *Let x_i $(i = 1, \dots, n)$ be the critical points of the function $K(x, 0)$ on the set $x = 1, 2, \dots, r$; let s_j $(j = 1, 2, \dots, m)$ be the critical points of the function $K(r, s)$ on the set $s = 0, 1, \dots, r-1$.*

If $P_0(x) > 0$ $(x = 1, 2, \dots, r-2)$, then:

a) $\operatorname{sgn} G(x, s) = \operatorname{sgn} K(r, 0) \cdot (-1)^{m+i-j+1}$ for $x = x_i + 1,\ x_i + 2, \dots, \min\{x_{i+1}, s+1\},\ s = s_j,\ s_j + 1, \dots, s_{j+1} - 1$ $(i = 0, 1, \dots, n;\ j = 0, 1, \dots, m;\ x_0 = 0,\ x_{n+1} = r,\ s_0 = 0,\ s_{m+1} = r-1)$, provided only that $G(x, s) \neq 0$;

b) $\operatorname{sgn} G(x, s) = \operatorname{sgn} G(s+1, x-1)$ for $(x, s) \in Q$.

If the condition $P_0(x) > 0$ $(x = 1, 2, \dots, r-2)$ is not fulfilled, Theorem 3 may turn out to be false. For example, if $P_0(x) < 0,\ P_1(x) < 0$ $(x = 0, 1, \dots, r-2)$, then $G(x, s) < 0$ for $x \leq s+1$ and $\operatorname{sgn} G(x, s) = (-1)^{x+s}$ for $x > s+1$ $((x, s) \in Q)$.

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CITED LITERATURE

1. N. V. Azbelev, Z. B. Tsalyuk, *Ukr. Mat. Zh.*, **10**, No. 1, 3 (1958).

2. N. Aronszajn, K. T. Smith, *Am. J. Math.*, **79**, No. 3, 611 (1957).

3. E. S. Chichkin, Final Scientific Conference of Kazan State University for 1960, Section of Mathematical Sciences, Cybernetics and Probability Theory, Mechanics, 1961, p. 40.

4. M. A. Skalkina, *Izv. Vyssh. Uchebn. Zaved., Matematika*, No. 6, 138 (1959).

5. A. L. Teptin, *Izv. Vyssh. Uchebn. Zaved., Matematika*, No. 5, 265 (1958).

6. A. L. Teptin, *Izv. Vyssh. Uchebn. Zaved., Matematika*, No. 3, 195 (1961).

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