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Abstract

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On the Effect of a Relaxation Boundary Layer

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The flow past a wedge by a supersonic stream of a weakly relaxing gas is considered. It is shown that nonequilibrium effects are substantial not only in a layer, several relaxation lengths ($u_0\tau_0$) thick, adjacent directly to the shock wave, but also in a region called the relaxation boundary layer, having a thickness of several relaxation lengths, adjoining the surface of the wedge and extending indefinitely downstream. An exact solution of the problem is given for vibrational relaxation.

The equations of motion of the gas with allowance for vibrational relaxation have the form:

$$\operatorname{div}\rho\mathbf{v} = 0; \qquad (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p; \qquad \frac{v^2}{2} + c_{pT} + E_i = h_0;$$

$$\frac{dE_i}{dt} = \frac{E_i(T) - E_i}{\tau}; \qquad p = \rho RT; \tag{1}$$

$$E_i(T_i) = \frac{R\theta_v}{e^{\theta_i/T_i} - 1};$$

where ρ is density; $\mathbf{v}$ is the velocity vector; p is pressure; T is the temperature of the translational degrees of freedom; $h_0,\ c_p,\ R,\ \theta_v$ are constants.

Consider the problem of flow past a wedge by a supersonic stream of a weakly relaxing gas (i.e., a flow in which the quantity $\overline{E_i}(T_0)$ is small); in this approximation the flow may be represented as the sum of an ideal flow, i.e., an oblique shock wave and a uniform stream (the quantities are denoted by the subscript 0), and a linear term, the equations for which can be obtained by linearizing system of equations (1) with respect to the parameter

$$\overline{E_i}(T_0) = \frac{E_i(T_0)}{c_{pT}0}.$$

Choosing the coordinate system as shown in Fig. 1, one can obtain the following system of equations for the perturbations (marked by a prime):

Fig. 1

Figure 1: Fig. 1

Fig. 1

$$\mu^2 \frac{\partial \bar{p}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \overline{E_i}}{\partial x} = 0; \qquad \frac{\partial \bar{p}}{\partial y} + \frac{\partial \bar{v}}{\partial x} = 0; \qquad \bar{p} + \bar{u} = F(y) \tag{2}$$

$$\left(\bar{p} = \frac{p'}{\rho_0 u_0^2};\ \bar{u} = \frac{u'}{u_0};\ \bar{v} = \frac{v'}{v_0};\ \overline{E_i} = \frac{E_i}{c_{pT}0};\ \text{the quantity } \mu^2 = M_0^2 - 1 \text{ is assumed positive}\right).$$

The expression for E_i, after integration of the relaxation equation with allowance for the boundary condition $E_i = 0$ at $y = x \operatorname{tg} \gamma$, takes the form:

$$\bar{E}_i = \bar{E}_i(T_0)\left(1 - e^{-\bar{x} + \bar{y} \operatorname{ctg} \gamma}\right) \qquad (\bar{x} = x / u_0 \tau_0;\ \bar{y} = y / u_0 \tau_0). \tag{3}$$

It is easy to see that

$$F'(y) = \frac{\partial \bar{u}}{\partial y} - \frac{\partial \bar{v}}{\partial x}; \tag{4}$$

from this follows the physical meaning of the function $F(y)$.

The boundary conditions under which the system of equations (2) must be solved have the form:

a) for $y = 0,\ x > 0,\ \bar{v} = 0$;

b) for $y = \operatorname{tg} \gamma \cdot x,\ x > 0,\ \bar{p} = \nu \bar{v},\ \bar{p} = k \bar{u}$,

where

$$\nu = \frac{2\Delta \operatorname{ctg} \gamma - (1 - \Delta)(\varkappa - 1) M_0^2 \sin \gamma \cos \gamma}{M_0^2 \varkappa (\sin^2 \gamma + \Delta \cos^2 \gamma) - 1 + \Delta \operatorname{ctg}^2 \gamma - (\varkappa - 1) M_0^2},$$

$$k = \frac{2\Delta - (1 - \Delta)(\varkappa - 1) M_0^2 \sin^2 \gamma}{M_0^2 \varkappa (1 - \Delta) \sin^2 \gamma - 1 - \Delta}, \qquad \Delta = \frac{p_\infty}{p_0}.$$

The general solution of system (2) may be represented in the form

$$\bar{p} = -\frac{n}{2\mu}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) e^{-\bar{x} + \bar{y} \operatorname{ctg} \gamma} - \frac{f_1(x + \mu y) + f_2(x - \mu y)}{2\mu},$$

$$\bar{v} = \frac{n}{2}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) e^{-\bar{x}+\bar{y}\operatorname{ctg}\gamma} + \frac{f_1(x+\mu y) - f_2(x-\mu y)}{2}; \tag{5}$$

$$n = \frac{\bar{E}_i(T_0)}{2\mu u_0 \tau_0}; \qquad \alpha = \frac{1-\mu\operatorname{tg}\gamma}{2\mu\operatorname{tg}\gamma\cdot u_0\tau_0}; \qquad \beta = -\frac{1+\mu\operatorname{tg}\gamma}{2\mu\operatorname{tg}\gamma u_0\tau_0}.$$

To determine the functions f_1 and f_2 we have the following system of functional equations:

$$n\left(\frac{1}{\beta} - \frac{1}{\alpha}\right) e^{-\bar{x}} + f_1(x) - f_2(x) = 0;$$

$$\frac{n}{\alpha}(1-\mu\nu) + \frac{n}{\beta}(1+\mu\nu) + f_1(k_1 y)(1+\mu\nu) + f_2(k_2 y)(1-\mu\nu) = 0.$$

Eliminating f_2, we obtain a functional equation for determining f_1:

$$A + \delta f_1(z) + \varepsilon f_1(mz) + Be^{-\bar{z}m} = 0;$$

$$A = \frac{n}{\alpha}\varepsilon + \frac{n}{\beta}\delta; \qquad \delta = 1+\mu\nu; \qquad \varepsilon = 1-\mu\nu; \tag{6}$$

$$B = n\varepsilon\left(\frac{1}{\beta} - \frac{1}{\alpha}\right); \qquad m = \frac{k_2}{k_1}; \qquad k_2 = \operatorname{ctg}\gamma - \mu; \qquad k_1 = \operatorname{ctg}\gamma + \mu.$$

It is natural to seek the solution of equation (6) in the form of the series

$$f_1(z) = D + \sum_{i=0}^{\infty} c_i e^{-\bar{z}m^i}.$$

Carrying out formally the necessary operations, we shall have:

$$f_1 = -\frac{A}{2} - \frac{B}{\delta}\sum_{i=0}^{\infty}\left(-\frac{\varepsilon}{\delta}\right)^i e^{-\bar{z}m^{i+1}};$$

$$f_2 = -\frac{A}{2} + \frac{B}{\varepsilon}e^{-\bar{z}} - \frac{B}{\delta}\sum_{i=0}^{\infty}\left(-\frac{\varepsilon}{\delta}\right)^i e^{-\bar{z}m^{i+1}}. \tag{7}$$

The series (7) converge everywhere faster than a geometric progression with ratio $(-\varepsilon/\delta)$. Thus,

$$\bar{p} = -\frac{n}{2\mu}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)e^{-\bar{x}+\bar{y}\,\mathrm{ctg}\,\gamma}+\frac{1}{2\mu}\left\{A-\frac{B}{\varepsilon}e^{-\bar{x}+\mu\bar{y}}+\frac{2B}{\delta}\sum_{i=0}^{\infty}\left(-\frac{\varepsilon}{\delta}\right)^{i}e^{-\bar{x}m^{i+1}}\,\mathrm{ch}\left(\mu\bar{y}m^{i+1}\right)\right\}. \tag{8}$$

It is easy to see that, for any $\bar{y}$, the quantity $\bar{p}$ tends, as $\bar{x}\to\infty$, to the constant limit

$$\bar{p}\big|_{\bar{x}=\infty} = \frac{A}{2\mu} = \bar{E}_i(T_0)\frac{1-\nu\,\mathrm{ctg}\,\gamma}{\mathrm{ctg}^2\gamma-\mu^2}.$$

We find the expression for $F(y)$ from the third equation of system (2), applied on the straight line $y = x\,\mathrm{tg}\,\gamma$; taking also into account the second condition b), we obtain:

$$F(y) = \frac{(1+k)B}{2\mu k\varepsilon}\left\{\mu\nu - e^{-\bar{y}(\mathrm{ctg}\,\gamma-\mu)}-2\sum_{i=1}^{\infty}\left(-\frac{\varepsilon}{\delta}\right)^{i}e^{-\bar{y}\,\mathrm{ctg}\,\gamma m^{i}}\,\mathrm{ch}\left(\mu\bar{y}m^{i}\right)\right\}. \tag{9}$$

It is easy to see that

$$F(0) = 0; \qquad F(\infty) = -\left(1+\frac{1}{k}\right)\bar{E}_i(T_0)\frac{\nu\,\mathrm{ctg}\,\gamma}{\mathrm{ctg}^2\gamma-\mu^2}.$$

Thus, applying the third equation of system (2) to the entire flow, we obtain that, as $x\to\infty$, in the immediate vicinity of the body being flowed around there is a vortex layer having a thickness of several relaxation lengths.

The expression for $\bar{v}$ has the form:

$$\bar{v} = \frac{n}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)e^{-\bar{x}+\bar{y}\,\mathrm{ctg}\,\gamma}+\frac{B}{2\varepsilon}\left\{e^{-\bar{x}+\mu\bar{y}}-2\sum_{i=1}^{\infty}\left(-\frac{\delta}{\varepsilon}\right)^{i}e^{-\bar{x}m^{i}}\,\mathrm{sh}\left(\mu\bar{y}m^{i}\right)\right\},$$

i.e., $\bar{v}$, for any $\bar{y}$, tends to zero as $\bar{x}\to\infty$.

Bringing together everything said about the structure of the flow, we see that the region of influence of relaxation consists of two regions (Fig. 2): first, the region A, directly adjacent to the shock wave and ending away from it at most within several relaxation lengths; and, second, the region B, situated directly at the boundary of the body being flowed around, having a thickness of several relaxation lengths and extending to infinity downstream. We shall call region B the relaxation boundary layer. In zone C there is equilibrium flow with a

Fig. 2

Figure 2: Fig. 2

fully excited oscillatory degree of freedom, corresponding to the equilibrium flow around the wedge.

Fig. 2

The hydrodynamic pressure changes only in zone A; across the relaxation boundary layer the pressure is unchanged. The remaining quantities characterizing the flow as $\bar{x} \to \infty$ are functions only of $\bar{y}$, so that the relaxation boundary layer does not change along the flow.

The velocity, temperature, and density will be, respectively,

$$\bar{u} = F(y) - \bar{p}\big|_{\bar{x}=\infty}; \qquad \bar{T} = -\bar{E}_i(T_0) - (\kappa - 1)M_0^2\,[F(y) - p\big|_{x=\infty}];$$
$$\bar{\rho} = \kappa M_0^2 \bar{p}\big|_{\bar{x}=\infty} - \bar{T} \qquad (\bar{\rho} = \rho'/\rho_0;\ \bar{T} = T'/T_0). \tag{10}$$

Figure 3 gives an example of a calculation for a relaxation boundary layer in the case of flow past a wedge with an angle of 40° by a gas stream with Mach numbers 5, 10, and 20 and $\kappa = 1.4$. Examination of Fig. 3 shows that for $M_\infty = 20$ the jump (i.e., $\varphi(0) - \varphi(\infty)$) of temperature, density, and velocity across the relaxation boundary layer is, respectively, $0.467\bar{E}_i(T_0)$; $-0.467\bar{E}_i(T_0)$; $-0.370\bar{E}_i(T_0)$, i.e., across the relaxation boundary layer the temperature increases in the direction toward the wall, the density decreases, and the velocity also decreases. The moduli of the ratios of these jumps to the values themselves as $\bar{y} \to \infty$ are, for temperature, density, and velocity, respectively, 0.29, 0.404, 0.79, i.e., the jumps of the quantities under consideration are of the order of the relative activation energy of the vibrational degree of freedom.

Fig. 3

Fig. 4

We next give expressions for the variation of the gas-dynamic quantities along the surface of the wedge:

$$\bar{p}\big|_{\bar{y}=0} = \frac{1}{\mu}\left\{\frac{n}{\beta}e^{-\bar{x}} + \frac{A}{2} + \frac{B}{\delta}\sum_{i=0}^{\infty}\left(-\frac{\varepsilon}{\delta}\right)^{i} e^{-\bar{x}m^{i+1}}\right\}, \tag{11}$$

$$\bar{u} = -\bar{p}; \qquad \bar{T} = (\kappa - 1)M_0^2\bar{p} - \bar{E}_i(T_0)(1 - e^{-\bar{x}}); \qquad \bar{\rho} = \kappa M_0^2\bar{p} - \bar{T}.$$

An example of the calculation is shown in Fig. 4 for a wedge with an angle of 40° and the conditions of Fig. 3. In practice, series of the type (11) converge so rapidly that for calculations it is sufficient to take the first 2-3 terms. In conclusion it should be said that the effect of the relaxation boundary layer noted in the present work is apparently inherent, to one degree or another, in all flows of thermodynamically nonequilibrium media.

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Note: Figure translations are in progress. See original paper for figures.

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