HYDROMECHANICS

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Abstract

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ON THE DYNAMICS OF A STRATIFIED FLUID

In the present note, under sufficiently general assumptions, an expression is obtained for the change of the potential vorticity (adiabatic invariant) in the presence of weak external influences (heat influx, friction forces) on a stratified fluid. The results may be applied to problems of sea dynamics and to the theory of atmospheric circulation.

A fluid is called stratified if the density ρ is a single-valued function of some physical characteristic of the medium σ and of the pressure p:

$$\rho = \rho(\sigma, p), \tag{1}$$

where the law of variation of σ with time for a fluid particle is assumed to be known. It is convenient to represent this law in the form

$$\frac{d\sigma}{dt} = \varepsilon q(x, y, z, t), \tag{2}$$

where the right-hand side εq describes the external influence on the medium; $\varepsilon = 1/T_0$ is a small parameter having the dimension of inverse time*. The solution corresponding to the value $\varepsilon = 0$ will be called the adiabatic approximation.

In problems of dynamical meteorology it is expedient to choose as σ the dimensionless entropy, defined for an ideal gas by the formula

$$\sigma = \frac{1}{\varkappa} \ln p - \ln \rho, \tag{3}$$

$S = c_v \sigma$ is the entropy in thermodynamic units; c_v is the heat capacity at constant volume, $\varkappa = c_p/c_v$ is the ratio of heat capacities. In problems of sea dynamics (motion of salt water) the compressibility of the medium can usually be neglected; taking the parameter

$$\beta = \left(\frac{\partial \rho}{\partial p}\right)_\sigma = \frac{1}{c^2} \tag{4}$$

(c is the speed of sound) equal to zero. In this case it is convenient to take for σ a quantity equal to the logarithm of the specific volume, i.e.

$$\sigma = -\lg \rho. \tag{5}$$

Formally, a stratified incompressible medium may be regarded as an "ideal gas" with the value $\varkappa = \infty$. The quantity σ in this case will also be called entropy.

The function $q(x, y, z, t)$ determines the influx of heat per unit mass of the fluid, referred to the period T_0 and divided by the temperature θ**.

The motion of the medium in a prescribed force field, characterized by the mass force $\mathbf{g}$, is described by the hydrodynamic equations

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p + \rho \mathbf{g}. \tag{6}$$

* In problems of dynamical meteorology, for $T_0 = 1/\varepsilon$ one should take some characteristic time of transformation of air masses.

** In the case where the change of density at fixed pressure is caused by a change of chemical composition (salinity, humidity), the function q may be treated as a certain "equivalent thermal influence."

Let us assume that the field $\mathbf{g}$ also depends on the small parameter and, in the adiabatic approximation ($\varepsilon = 0$), reduces to a potential field (the gravitational field). Accordingly,

$$\mathbf{g} = -\operatorname{grad} \Phi + \varepsilon \mathbf{f}, \tag{7}$$

where, generally speaking, $\operatorname{rot} \mathbf{f} \neq 0$. Within this scheme one can take viscosity forces into account by putting

$$\varepsilon \rho \mathbf{f} = \mu \cdot \Delta \mathbf{v}, \tag{8}$$

whence

$$\mathbf{f} = \Sigma \cdot \Delta \mathbf{v}, \tag{9}$$

where

$$\Sigma = \frac{\mu T_0}{\rho} \tag{10}$$

is "a certain effective cross-section" describing the integral action of viscosity over the period T_0*.

Using the representation (7), we write the equations of motion (6) in the following form:

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\operatorname{grad} p - \operatorname{grad}\Phi + \varepsilon\mathbf{f}. \tag{11}$$

To the vector equation (11) one must add the continuity equation

$$\frac{d\rho}{dt} + \rho \operatorname{div}\mathbf{v} = 0, \tag{12}$$

as well as the equation of state (1) and the equation of heat influx (2). Taken together, equations (11), (12), (1), and (2) form a complete system with respect to the unknown quantities $\mathbf{v}, \rho, p, T$.

As is known ([1,2]), the quantity $I = (\vec{\Omega}\operatorname{grad}\sigma/\rho)$, where $\vec{\Omega} = \operatorname{rot}\mathbf{v}$, is an adiabatic invariant, i.e., for $\varepsilon = 0$

$$\frac{d}{dt}\left(\frac{\vec{\Omega}\operatorname{grad}\sigma}{\rho}\right) = 0. \tag{13}$$

Let us now compute the derivative dI/dt in the general case. Applying the operation rot to the equations of motion (11), we obtain the well-known equation of A. A. Friedman ([3])

$$\frac{d\vec{\Omega}}{dt} - (\vec{\Omega}\nabla)\mathbf{v} + \vec{\Omega}\operatorname{div}\mathbf{v} = \frac{1}{\rho^2}[\operatorname{grad}\rho, \operatorname{grad} p] + \varepsilon\operatorname{rot}\mathbf{f}. \tag{14}$$

Let us now apply the differentiation operator in the direction of the vector $\vec{\Omega}$ to the total derivative $d\sigma/dt$

$$(\vec{\Omega}\nabla)\left(\frac{\partial}{\partial t} + (\mathbf{v}\nabla)\right)\sigma = \frac{\partial}{\partial t}(\vec{\Omega}\nabla)\sigma - \frac{d\vec{\Omega}}{dt}\nabla\sigma + (\mathbf{v}\nabla)(\vec{\Omega}\nabla)\sigma - (\mathbf{v}\nabla\vec{\Omega})\nabla\sigma +$$

$$+(\vec{\Omega}\nabla)\mathbf{v}\cdot\nabla\sigma = \frac{d}{dt}(\vec{\Omega}\nabla)\sigma - \frac{d\vec{\Omega}}{dt}\nabla\sigma + (\vec{\Omega}\nabla)\mathbf{v}\cdot\nabla\sigma. \tag{15}$$

It follows from this, on the basis of equation (2), that

$$\frac{d}{dt}(\vec{\Omega}\nabla)\sigma - \frac{d\vec{\Omega}}{dt}\nabla\sigma + (\vec{\Omega}\nabla)\mathbf{v}\cdot\nabla\sigma = \varepsilon(\vec{\Omega}\nabla)q. \tag{16}$$

* The quantity Σ has the dimension of area. Taking, for the conditions of the Earth' s atmosphere, $\rho = 10^{-3}\,\mathrm{g}\cdot\mathrm{cm}^{-3}$, $\mu = 2\cdot 10^{-4}\,\mathrm{g}\cdot\mathrm{cm}^{-1}\cdot\mathrm{sec}^{-1}$, $T_0 = 10^6$ sec. (12 days), we obtain $\Sigma = 2\cdot 10^5$ cm^2, which corresponds to the area of a circle with diameter about 5 m.

Multiplying (14) scalarly by $\nabla\sigma = \operatorname{grad}\sigma$ and adding with (16), we obtain

$$\frac{d}{dt}(\vec{\Omega}\operatorname{grad}\sigma)+(\vec{\Omega}\operatorname{grad}\sigma)\operatorname{div}\mathbf{v} = \frac{1}{\rho^2}([\operatorname{grad}\rho, \operatorname{grad}p], \operatorname{grad}\sigma)+\varepsilon(\operatorname{rot}\mathbf{f}\cdot\operatorname{grad}\sigma)+\varepsilon(\vec{\Omega}\operatorname{grad}q). \tag{17}$$

The quantity $\chi = \vec{\Omega}\operatorname{grad}\sigma$ may be interpreted as the volume density of a certain "vortex charge." By virtue of the equation of state (1), the triple product on the right-hand side vanishes identically, and we obtain

$$\frac{d}{dt}\chi + \chi\operatorname{div}\mathbf{v} = \varepsilon(\vec{\Omega}\operatorname{grad}q) + \varepsilon(\operatorname{rot}\mathbf{f}\cdot\operatorname{grad}\sigma). \tag{18}$$

Let us consider the motion of an arbitrary fluid volume V; from equation (18) it follows that

$$\frac{d}{dt}\iiint\limits_V \chi\,dx\,dy\,dz = \iiint\limits_V \left[\varepsilon(\vec{\Omega}\operatorname{grad}q) + \varepsilon(\operatorname{rot}\mathbf{f}\cdot\operatorname{grad}\sigma)\right]dx\,dy\,dz. \tag{19}$$

For $\varepsilon = 0$ we obtain the theorem on the conservation of the total "vortex charge" $\iiint\limits_V \chi\,dV$ during the motion of a fluid volume. The right-hand side of the equation denotes the volume density of vorticity sources caused by nonadiabatic processes (heat exchange, friction).

Using the law of conservation of mass

$$\frac{d}{dt}\iiint\limits_V \rho\,dx\,dy\,dz = 0,$$

which in differential form is described by the continuity equation (12), it is easy to show that the quantity

$$\frac{\chi}{\rho} = I = \left(\frac{\vec{\Omega}\operatorname{grad}\sigma}{\rho}\right), \tag{20}$$

which has the meaning of "specific vorticity" —a vortex charge calculated per unit mass—satisfies the equation

$$\frac{dI}{dt} = \varepsilon\frac{(\vec{\Omega}\,\mathrm{grad}\,q)}{\rho} + \varepsilon\frac{(\mathrm{rot}\,\mathbf{f}\cdot\mathrm{grad}\,\sigma)}{\rho}. \tag{21}$$

and for $\varepsilon = 0$ (adiabaticity) is an absolute invariant. A quantity analogous to the invariant I was introduced by Rossby[(1)] in the study of processes in a two-dimensional rotating fluid and was called by him potential vorticity. The proof of the invariance of I in the three-dimensional case for adiabatic atmospheric motions was given by Charney[(2)], and also by Arakawa[(4)]. The dynamical invariants obtained by the author and A. S. Monin[(5,6)] in the linearized theory of atmospheric processes may be regarded as a certain approximate expression for specific vorticity, obtained as a result of the process of linearization about a relative basic state. In the theory of short-range weather prediction the quantity I plays a fundamental role[(7)].

In problems of the theory of long-range weather prediction and the general circulation of the atmosphere, when considering time intervals comparable with T_0, nonadiabatic changes in the field of specific vorticity must be taken into account by means of equation (21).

Let us consider some special cases. Suppose that frictional forces are absent, so that $\mathrm{rot}\,\mathbf{f} = 0$, and that the thermal factor q, although different from zero, does not cause a change in the vorticity field. Such a field of heat sources may naturally be called passive. From (21) it follows that in this case $(\vec{\Omega}\,\mathrm{grad}\,q) = 0$,

i.e., the density of entropy sources must be constant along any vortex line. We shall call two different thermal actions, described by q_1 and q_2, dynamically equivalent if they produce one and the same change in specific vorticity. The condition for the dynamical equivalence of two distributions q_1 and q_2 is the constancy of their difference along vortex lines. Suppose now that both the thermal factor $(\vec{\Omega}\,\mathrm{grad}\,q)$ and the dynamical factor $(\mathrm{rot}\,\mathbf{f}\cdot\mathrm{grad}\,\sigma)$ of the change in vorticity are nonzero, but that their total action is zero ($dI/dt = 0$). We shall call such a field of external actions compensated*. From formula (21) one obtains the compensation condition, which may be written in the following form

$$\mathrm{div}(q\vec{\Omega} + \sigma\,\mathrm{rot}\,\mathbf{f}) = 0. \tag{22}$$

Thus, under the conditions of a compensated field of external actions, the vector field $\mathbf{N} = q\vec{\Omega} + \sigma\,\mathrm{rot}\,\mathbf{f}$ is solenoidal.

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* Under real conditions in the atmosphere and ocean, such compensation may occur when considering a certain mean circulation regime.

Note: Figure translations are in progress. See original paper for figures.