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Abstract

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MATHEMATICS

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ON SUBHARMONIC FUNCTIONS OF COMPLETELY REGULAR GROWTH IN MULTIDIMENSIONAL SPACE

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In this paper, the known theorems on the connection between the growth indicator and the distribution of zeros of entire functions of finite order ρ and completely regular growth (^{3}, Chs. II, III) are carried over to functions subharmonic in the whole space of m dimensions. Along the way, the analytic properties of the indicator of a function subharmonic in the whole space or in a certain cone are investigated. This part of the paper adjoins the author' s preceding note (^{5}).

Introduce in the space E_m of m dimensions the spherical system of coordinates by the formulas:

$$\begin{aligned} x_1 &= r\sin\theta_0\sin\theta_1 \dots \sin\theta_{m-2}, \\ x_2 &= r\cos\theta_0\sin\theta_1 \dots \sin\theta_{m-2}, \\ &\dots \\ x_m &= r\cos\theta_{m-2} \end{aligned} \tag{1}$$

$$(0 \le r < \infty; \quad 0 \le \theta_0 < 2\pi; \quad 0 \le \theta_i \le \pi; \quad i = 1, 2, \dots, m-2).$$

We shall use the following notation: S_R is the sphere of radius R; K_R is the ball of radius R; dV is an element of the surface of the unit sphere S_1; dl is an element of the surface of an $(m-2)$-dimensional contour on S_1; $\mathbf{x}$ is a vector from S_1; any vector $\mathbf{r} \in E_m$ with origin at zero will be written in the form $\mathbf{r} = r\mathbf{x}$, where $r = |\mathbf{r}|$, $\mathbf{x} = \mathbf{r} \cdot r^{-1}$ $(\in S_1)$, and the value of a function u at the end of this vector as $u(r\mathbf{x})$.

Denote by L the following differential operator of second order:

$$L = \sum_{i=0}^{m-2} \frac{1}{\Pi} \frac{\partial}{\partial \theta_i} \cdot \frac{\Pi}{\Pi_i} \frac{\partial}{\partial \theta_i} + \rho(\rho + m - 2),$$

where

$$\Pi = \prod_{j=0}^{m-2} \sin^j \theta_j, \qquad \Pi_i = \prod_{j=i+1}^{m-2} \sin^2 \theta_j, \qquad \rho \geq 0.$$

Let an $(m-2)$-dimensional contour l be given on the sphere S_1 by the equations

$$\theta_i = \theta_i(\lambda_1, \dots, \lambda_{m-2}), \qquad i = 0, 1, \dots, m-2.$$

Denote by $\partial/\partial n$ the derivative along the normal to this contour:

$$\frac{\partial}{\partial n} = \sum_{k=0}^{m-2} \frac{\sqrt{\Pi}}{\Pi_k} \cdot D_k \left[\sum_{j=0}^{m-2} \Pi_j^{-2} D_j^2 \right]^{-1/2} \frac{\partial}{\partial \theta_k},$$

where D_k are the Jacobians:

$$D_k = (-1)^k D(\theta_0, \dots, \theta_{k-1}, \theta_{k+1}, \dots, \theta_{m-2}) / D(\lambda_1, \dots, \lambda_{m-2}),$$

$$k = 0, 1, \dots, m-2.$$

In this notation, for any sufficiently smooth functions, i.e. having, for example, continuous second derivatives, u and v, defined on the sphere S_1, and for an arbitrary domain V on this sphere bounded by a smooth contour l, the generalized Green formula holds (([1]), p. 235):

$$\int_V (uLv - vLu)\, dV = \int_l \left(u \frac{\partial}{\partial n} v - v \frac{\partial}{\partial n} u \right) dl. \tag{2}$$

We shall denote by W an arbitrary domain in S_1, bounded by a smooth contour l, in which the Dirichlet problem corresponding to the operator L is uniquely solvable. Denote by $Y_W^\varphi(x)$ the solution of the equation

$$LY = 0 \tag{3}$$

with boundary conditions

$$Y_W^\varphi(x)\big|_{x \in l} = \varphi(x).$$

Definition. A function $h(x)$ is called **subspherical** if it satisfies the following conditions: 1) it is upper semicontinuous in S_1; 2) it is finite on an everywhere dense set of points $x \in S_1$; 3) if $\varphi(x)$ is an arbitrary continuous function defined on l—the boundary of the domain W—and satisfying the inequality

$$\varphi(x) \geq h(x)\big|_{x \in l},$$

then the inequality

$$Y_W^{\varphi}(x) \geq h(x)$$

holds throughout the domain W.

As was shown in (^{5}), the concept of subsphericity is a generalization of the concept of trigonometric convexity and is analogous to the concepts of subharmonicity and convexity.

Equation (3) has a fundamental solution $\Phi(\sin\theta)$, where θ is the angle between the vector $\vec{\xi}$, determining the point on S_1 at which Φ has a singularity, and the vector $x \in S_1$. One can choose Φ so that the condition

$$\lim_{\varepsilon \to 0} \int_{\theta_{m-2}=\varepsilon} \frac{\partial}{\partial n} \Phi \, dl = 1$$

is fulfilled.

For a subspherical function there is a representation analogous to the F. Riesz representation for a subharmonic function.

Theorem 1. *If $h(x)$ is a subspherical function, then for any closed domain $V \subset S_1$ there is a representation*

$$h(x) = Y(x) + \int_{\vec{\xi} \in V} \Phi \, d\Delta_{\vec{\xi}},$$

where $Y(x)$ satisfies equation (3) in the domain V; $\Delta_{\vec{\xi}}$ is an additive, nonnegative, upper semicontinuous set function (Radon measure), and the integral is understood in the Stieltjes sense.

We shall denote the measure of a set $V \subset S_1$ by $\Delta(V)$. For a twice continuously differentiable subspherical function $h(x)$, the inequality $Lh \geq 0$ holds. This follows from the relation

$$h(x) - Y_W^h(x) = -\int_W Lh \cdot G(x, \vec{\xi}) \, dV,$$

obtained from formula (2) by substituting

$$u = h - Y_W^h, \qquad v = G(x, \vec{\xi}),$$

where $G(x, \vec{\xi})$ is the Green function of the operator L for the domain W.

Applying the operator L to the function

$$Y(x) = h(x) - \int_{\vec{\xi} \in V} \Phi L h \, dV,$$

we see that $LY = 0$. Denoting $d\Delta_{\vec{\xi}} = Lh\, dV$, we obtain the assertion of the theorem for a twice continuously differentiable function $h(x)$.

The passage to the general case is based on the following lemma, which we state without proof.

Lemma. For every subspherical function $h(x)$ there exists a sequence of twice continuously differentiable functions $\{h_n\}_0^\infty$ such that $h_n \geq h$ and $\{h_n\}_0^\infty$ converges to h at every point, and moreover, for every domain $V \subset S_1$ the condition

$$\int_V L h_n \, dV < \infty$$

is satisfied.

The completion of the proof of the theorem is carried out by the standard method (see, for example, (^{4}), pp. 156-159).

Let now $h(x)$ be the indicator of a function $u(rx)$, subharmonic in the whole space, of finite order ρ and normal type (see (^{5})), i.e.

$$h(x) = \overline{\lim_{r \to \infty}} \, u(rx) \, r^{-\rho}.$$

In (^{5}) it was shown that $h(x)$ has properties 2) and 3) from the definition of a subspherical function. Its upper semicontinuous regularization $\tilde{h}(x)$ is therefore a subspherical function $(\tilde{h}(x) = \overline{\lim}_{x' \to x} h(x'))$.

Theorem 2. The indicator $h(x)$ can differ from its regularization only on such a set $G \subset S_1$ that its intersection with any piecewise smooth contour l has zero measure on this contour.

The proof is omitted. We note only that it is based on A. Cartan' s theorem (^{2}) on the upper envelope of a family of subharmonic functions.

Denote by γ an arbitrary subspace of $m-2$ dimensions of the space E_m, and by ψ the subspace orthogonal to it—a plane. Choose the coordinate system (1) so that the coordinate axes x_1 and x_2 lie in the plane ψ. Then γ is defined by the equation $\theta_1 = 0$. The equation $\theta_0 = 0$ defines the "half-hyperplane" spanned by γ and the positive ray of the coordinate axis x_2.

Definition. The γ-projection of a point Q with coordinates $(r, \theta_0, \theta_1, \dots, \theta_{m-2})$ onto the "half-hyperplane" $\theta_0 = 0$ is the point Q^0_γ with coordinates $(r, 0, \theta_1, \dots, \theta_{m-2})$. The set of γ-projections of points of a set $C \subset E_m$ is called the γ-projection of the set C and is denoted C^0_γ. The measure of the set C^0_γ on the hyperplane $x_1 = 0$ is called the γ-measure of the set C. It does not change under rotation of the coordinate axes x_1 and x_2 in the plane ψ and is denoted $m_\gamma(C)$.

For a set $C \subset E_m$ define the quantities $\rho_R(C)$ and $\omega_R(C)$ by the equalities

$$\rho_R(C) = \sup_\gamma m_\gamma(C \cap K_R); \qquad \omega_R(C) = \text{mes}(C \cap S_R),$$

where mes denotes measure on the sphere S_R.

Definition. A set C is called a C_0-set if it satisfies the condition

$$\lim_{R\to\infty} \{\rho_R(C) + \omega_R(C)\} R^{-m+1} = 0.$$

Definition. A subharmonic function in the whole space $u(rx)$ of order ρ and of normal type is called a **function of completely regular growth** if the function

$$h_r(x) = u(rx) r^{-\rho}$$

converges uniformly to $h(x)$, the indicator of the function $u(rx)$, under the condition that $rx \to \infty$ in such a way that the endpoint of the vector rx does not belong to some C_0-set.

Let now $\mu(K^V_R)$ denote the mass corresponding to the subharmonic function $u(rx)$, contained in the sector K^V_R $(0 \le r < R,\ x \subset V \subset S_1)$; $h(x)$ is the indicator of $u(rx)$; Δ_x is the measure constructed by Theorem 1 for the regularized indicator $\tilde{h}(x)$. We shall call the set V **regular with respect to Δ_x** if the condition

$$\Delta(V) = \sup_{\widetilde{V} \subset V} \Delta(\widetilde{V})$$

is satisfied.

Theorem 3. *In order that a subharmonic function in the whole space $u(rx)$ of nonintegral order ρ and of normal type be a function of completely regular growth, it is necessary and sufficient that the limit*

$$\lim_{R\to\infty} \mu(K_R^V)R^{-\rho} = \frac{1}{\rho+m-2}\Delta(V)$$

exist for every set V *regular with respect to* Δ_x.

An analogous assertion holds for subharmonic functions of integral order.

These theorems solve one of the problems posed by B. Ya. Levin in his report at the Third All-Union Mathematical Congress in 1956.

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