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Abstract

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ON THE EIGENVALUES AND SINGULAR VALUES OF THE SUM AND PRODUCT OF LINEAR OPERATORS

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V. B. Lidskii ([1]) proved a theorem establishing, in a visual geometric form, a connection between the eigenvalues of the sum of Hermitian matrices and the eigenvalues of the summands, and also solved the analogous problem for the product of positive definite matrices*.

In the present note we give several propositions on the singular values of matrices (Sec. 2), analogous to the theorems of V. B. Lidskii, and also extensions of V. B. Lidskii' s theorems and of the indicated propositions to the infinite-dimensional case (Secs. 3–5). The proofs of the theorems given below are based on certain assertions of a geometric character (Sec. 1), established with the aid of the theorem of M. G. Krein and D. P. Milman ([3]) on the extreme points of a convex bicompact set, and also on the ideas of Weyland' s work ([4]), which indicated a new proof of V. B. Lidskii' s theorem on the eigenvalues of the sum of Hermitian matrices.

1. Let R^n be an n-dimensional Euclidean space, c_0 (respectively l_1) the space of real sequences tending to zero (respectively absolutely summable), $\alpha = \{\alpha_j\}_1^\infty$, $\hat{c}_0$ and $\hat{l}_1$ the same spaces, but consisting of sequences $\alpha = \{\alpha_j\}_{-\infty}^{\infty\prime}$, where the prime denotes omission of the index $j = 0$.

If $\alpha = \{\alpha_j\}$ and $\beta = \{\beta_j\}$ are vectors of the space R^n (or c_0, or $\hat{c}_0$) and

$$\sup_{j_1<\cdots<j_k} \sum_{m=1}^{k} \alpha_{j_m} \leq \sup_{j_1<\cdots<j_k} \sum_{m=1}^{k} \beta_{j_m} \quad (k = 1, 2, ...),$$

then we shall write $\alpha \ll \beta$. If $\alpha, \beta \in \hat{l}_1$, $\alpha \ll \beta$, $-\alpha \ll -\beta$, and

$$\sum_{j=-\infty}^{\infty\prime} \alpha_j = \sum_{j=-\infty}^{\infty\prime} \beta_j,$$

then we shall write $\alpha < \beta$.

Lemma 1. Let $\beta \in R^n$ and $\beta_j \geq 0$ $(j = 1, 2, \dots, n)$. For a vector $\alpha \in R^n$ the condition

$$(|\alpha_1|, |\alpha_2|, \dots, |\alpha_n|) \ll (\beta_1, \beta_2, \dots, \beta_n)$$

is fulfilled if and only if α is contained in the convex hull of all possible vectors of the form

$$(\varepsilon_1 \beta_{j_1}, \varepsilon_2 \beta_{j_2}, \dots, \varepsilon_n \beta_{j_n}),$$

where $\varepsilon_k = \pm 1$ and $j_1, j_2, \dots, j_n$ is an arbitrary permutation of the numbers $1, 2, \dots, n$.

Lemma 2. Let $\beta \in c_0$ and $\beta_j \geq 0$ $(j = 1, 2, \dots)$. For a real vector $\alpha = \{\alpha_j\}_1^\infty$ the condition

$$(|\alpha_1|, |\alpha_2|, \dots) \ll (\beta_1, \beta_2, \dots)$$

is fulfilled if and only if α is contained in the convex hull, closed in the norm of c_0, of all possible vectors of the form

$$(\varepsilon_1 \beta_{j_1}, \varepsilon_2 \beta_{j_2}, \dots), \tag{1}$$

where $\varepsilon_k = \pm 1$ and $j_1, j_2, \dots$ is an arbitrary permutation of the natural sequence.

Remark 1. If $\beta \in l_p$ $(p \geq 1)$, then the convex hulls of all possible vectors of the form (1), closed in the norm of c_0 and in the norm of l_p, coincide.

Lemma 3. If $\beta \in \hat{l}_1$ (respectively $\hat{c}_0$), then the convex hull, closed in the norm of $\hat{l}_1$ (respectively $\hat{c}_0$), of all possible vectors obtained

* These problems first arose in the investigations of I. M. Gelfand and M. A. Naimark on the theory of representations of groups; their solution, based on group methods, was indicated by F. A. Berezin and I. M. Gelfand ([2]).

by permutations of the coordinates of the vector β, coincides with the set of all vectors α such that $\alpha \prec \beta$ (respectively, $\alpha \ll \beta$ and $-\alpha \ll -\beta$).

Remark 2. If $\beta \in \hat{l}_p$ $(p > 1)$, then the convex closures, closed in the norm $\hat{c}_0$ and in the norm $\hat{l}_p$, of all possible vectors obtained by permutations of the coordinates of the vector β, coincide.

This lemma (in the part concerning $\hat{l}_1$) is an infinite-dimensional analogue of a proposition of Rado (5) (which can also be obtained by combining earlier results of Hardy, Littlewood, and Pólya (6), p. 66, and Birkhoff (7)).

2. Let A be a complex matrix of order n. By $s_j(A)$ $(j = 1, 2, \dots, n)$ we denote the **singular numbers** of the matrix A, i.e. the square roots, arranged in decreasing order, of the eigenvalues of the matrix $A^* A$.

The following proposition is established with the aid of certain results of Wielandt (4) and Mirsky (8) and Lemma 1.

Theorem 1. *If A and B are matrices of order n and $C = A + B$, then the vector*

$$(s_1(C),\, s_2(C),\, \dots,\, s_n(C))$$

is contained in the convex hull of all possible vectors of the form

$$(s_1(B) + \varepsilon_1 s_{j_1}(A),\, s_2(B) + \varepsilon_2 s_{j_2}(A),\, \dots,\, s_n(B) + \varepsilon_n s_{j_n}(A)),$$

where $\varepsilon_k = \pm 1$, and $j_1, j_2, \dots, j_n$ is an arbitrary permutation of the numbers $1, 2, \dots, n$.

Corollary 1. *If, under the hypotheses of Theorem 1,*

$$s_1(A) + s_2(A) < s_k(B) - s_{k+1}(B) \qquad (k = 1, 2, \dots, n-1),$$

then all the numbers $s_j(C)$ $(j = 1, 2, \dots, n)$ are distinct.

Theorem 2. *If A and B are nonsingular matrices of order n and $C = AB$, then the vector*

$$(\ln s_1(C),\, \ln s_2(C),\, \dots,\, \ln s_n(C))$$

is contained in the convex hull of all possible vectors of the form

$$(\ln s_1(B) + \ln s_{j_1}(A),\, \ln s_2(B) + \ln s_{j_2}(A),\, \dots,\, \ln s_n(B) + \ln s_{j_n}(A)),$$

where $j_1, j_2, \dots, j_n$ is an arbitrary permutation of the numbers $1, 2, \dots, n$.

This theorem follows from results of Amir-Moéz [(9)], Theorem 3.10, and Rado [(5)].

3. Following [(10)], by $\mathfrak{S}_\infty$ we denote the set of all linear completely continuous operators acting in a separable Hilbert space $\mathfrak{H}$, and by $\mathfrak{S}_1$ the subset of $\mathfrak{S}_\infty$ consisting of operators with finite trace.

If A is a self-adjoint completely continuous operator, then by

$$\lambda(A) = \{\lambda_j(A)\}_{-\infty}^{\infty}$$

we shall denote the complete system of eigenvalues of the operator A, in which they are numbered (with multiplicities taken into account) by indices from $-\infty$ to ∞, with the omission of the index $j = 0$, and moreover

$$\lambda_j(A) \geq 0, \quad \lambda_{-j}(A) \leq 0, \quad \lambda_j(A) \geq \lambda_{j+1}(A), \quad \lambda_{-j}(A) \leq \lambda_{-j-1}(A) \quad (j = 1, 2, \dots)^*.$$

Lemma 4. *If $H \in \mathfrak{S}_\infty$ and $T \in \mathfrak{S}_1$ are self-adjoint operators, then*

$$\sum_{j=-\infty}^{\infty} (\lambda_j(H + T) - \lambda_j(H)) = \operatorname{Sp} T,$$

and the series converges absolutely.

With the aid of Lemmas 3 and 4 one proves the following proposition, which is a generalization of a theorem of V. B. Lidskii [(1)].

Theorem 3. *Let A be a self-adjoint operator in $\mathfrak{S}_\infty$ (respectively $\mathfrak{S}_1$), B a self-adjoint operator in $\mathfrak{S}_\infty$, and $C = A + B$. Then,*

* The term "complete" has here not quite its usual meaning, since if the operator A has an infinite number of both positive and negative eigenvalues, then the sequence $\{\lambda_j(A)\}_{-\infty}^{\infty}$ will consist only of them, and, consequently, the eigenvalues equal to zero will not enter it, even if such exist; if the operator A has a finite number n (≥ 0) of positive (negative) eigenvalues, then we put $\lambda_{n+j}(A) = 0,\ j = 1, 2, \ldots$ $(\lambda_{-n-j}(A) = 0,\ j = 1, 2, \ldots)$, irrespective of whether the number 0 is an eigenvalue of the operator A or not.

the vector $\lambda(C) - \lambda(B)$ is contained in the convex closed (in the norm $\hat{c}_0$ (respectively $\hat{l}_1$) hull of all possible vectors obtained by permuting the coordinates of the vector $\lambda(A)$.

4. If $A \in \mathfrak{S}_\infty$, then by $s(A) = \{s_i(A)\}_1^\infty$ is denoted the sequence of **singular numbers** of the operator A, i.e., the sequence of eigenvalues of the operator $(A^*A)^{1/2}$, numbered in decreasing order with their multiplicities taken into account.

The following theorem, proved with the aid of Lemma 2, is a generalization of Theorem 1, and also of one result of Fan Ky ([11], Theorem 5).

Theorem 4. If A and B are completely continuous operators and $C = A + B$, then the vector $s(C) - s(B)$ is contained in the convex closed (in the norm c_0) hull of all possible vectors of the form $(\varepsilon_1 s_{j_1}(A), \varepsilon_2 s_{j_2}(A), \ldots)$, where $\varepsilon_k = \pm 1$, and $j_1, j_2, \ldots$ is any permutation of the natural sequence.

With the aid of Fan Ky' s theorem ([11], Theorem 4) one obtains

Corollary 2. Let $\|A\|$ be some unitarily invariant norm and $\Phi(s_1, s_2, \ldots)$ the symmetric norming function generating it (in the sense of Neumann and Schatten [12]). Then, for any $A, B \in \mathfrak{S}_\infty$,

$$\|A - B\| \geq \Phi(s_1(A) - s_1(B),\ s_2(A) - s_2(B), \ldots).$$

For the finite-dimensional case this assertion was obtained by Mirsky [8].

5. **Theorem 5.** If A and B are completely continuous operators and $C = AB$, then for every natural k and any distinct natural numbers $j_1, j_2, \ldots, j_k$

$$\prod_{n=1}^{k} s_{j_n}(C) \leq \prod_{n=1}^{k} s_n(A) \prod_{n=1}^{k} s_{j_n}(B).$$

This theorem is a generalization of Horn' s theorem [13].

Theorem 2 admits a generalization, for the formulation of which we shall need the following notation. If $T \in \mathfrak{S}_\infty$ and $A = I + T$, put

$$s_j(A) = \left(1 + \lambda_j(T + T^* + T^*T)\right)^{1/2} \quad (j = \pm 1, \pm 2, \ldots).$$

Theorem 6. Let $T_1 \in \mathfrak{S}_\infty$ (respectively $T_1 \in \mathfrak{S}_1$) and $T_2 \in \mathfrak{S}_\infty$, the operators $A = I + T_1$ and $B = I + T_2$ vanish only at zero, and let $C = AB$. Then the vector $\{\ln s_j(C) - \ln s_j(B)\}_{-\infty}^{\infty}$ is contained in the convex closed (in the norm $\hat{c}_0$ (respectively $\hat{l}_1$) hull of all possible vectors obtained by permuting the coordinates of the vector $\{\ln s_j(A)\}_{-\infty}^{\infty}$.

There also hold assertions, analogous to Theorems 5 and 6, about the eigenvalues of the product of positive operators (the second of them is an infinite-dimensional analogue of Theorem 4 of V. B. Lidskii ([1])).

Let us note in conclusion that, in the assertions of all the theorems presented, one may interchange the operators A and B (for Theorems 1, 3, and 4 this is obvious, while for Theorems 2, 5, and 6 it follows from the coincidence of the singular numbers of the operators A and A^*).

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Note: Figure translations are in progress. See original paper for figures.