Reports of the Academy of Sciences of the USSR

MATHEMATICS

1962

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1962. Volume 145, No. 1

MATHEMATICS

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ON THE SIMULTANEOUS APPROXIMATION OF ALMOST-PERIODIC FUNCTIONS AND THEIR DERIVATIVES

(Presented by Academician V. I. Smirnov on 16 II 1962)

1. Let $W^{(r)}$ be the class of all functions bounded on $(-\infty, \infty)$ that have a derivative of order r bounded on the entire real axis; let $P^{(r)}$ and $P_{2\pi}^{(r)}$ be, respectively, the classes of all periodic and all 2π-periodic functions from $W^{(r)}$. We shall say that a uniformly almost-periodic function $f(x)$ belongs to the class A_s if $N_f(x, x+1) = O(1)$, where $N_f(x, x+1)$ is the number of Fourier exponents of the function $f(x)$ on the interval $(x, x+1)$. (The structural characteristic of the class A_s is given in ([5]).) Denote by $W_s^{(r)}$ the intersection of the classes $W^{(r)}$ and A_s. Obviously, $W_s^{(r)} \supset P_{2\pi}^{(r)}$. Set:

$$C_{\sigma,r}(f) = \mathrm{Inf}_{g_\sigma(x)} \max_{0\le k\le r} \frac{\mathrm{Sup}_x\, |f^{(k)}(x) - g_\sigma^{(k)}(x)|}{E_\sigma(f^{(k)})},$$

where $g_\sigma(x)$ is an entire function of degree $\le \sigma$ and

$$E_\sigma(f) = \mathrm{Inf}_{g_\sigma(x)}\, \mathrm{Sup}_x\, |f(x) - g_\sigma(x)|;$$

$$C^*_{n,r}(f) = \mathrm{Inf}_{T_n(x)} \max_{0\le k\le r} \frac{\mathrm{Sup}_x\, |f^{(k)}(x) - T_n^{(k)}(x)|}{E_n^*(f^{(k)})},$$

where

$$T_n(x) = \sum_{\nu=0}^{n} a_\nu \cos \nu x + b_\nu \sin \nu x$$

and

$$E_n^*(f) = \mathrm{Inf}_{T_n(x)}\,\mathrm{Sup}_x\,|f(x) - T_n(x)|.$$

Let

$$C_{\sigma,r}(W^{(r)}) = \mathrm{Sup}_{f\in W^{(r)}}\,C_{\sigma,r}(f);$$

analogously the quantities $C_{\sigma,r}(W_s^{(r)})$, $C_{\sigma,r}(P^{(r)})$, $C_{n,r}^*(P_{2\pi}^{(r)})$ are defined.

A. F. Timan ([1]) showed that, as $r \to \infty$, uniformly with respect to all $\sigma > 0$, the following asymptotic equality holds:

$$C_{\sigma,r}(W^{(r)}) = \frac{4}{\pi^2}\ln(r+1) + O(\ln\ln\ln r). \tag{1}$$

A. L. Garkavi ([2]) obtained the asymptotic formula

$$C_{n,r}^*(P_{2\pi}^{(r)}) = \frac{4}{\pi^2}\ln(p+1) + O(\ln\ln\ln p), \tag{2}$$

where $p = \min\{n, r\}$.

It follows from the theorems given below that the result of A. L. Garkavi can be extended to the class $W_s^{(r)}$ of almost-periodic functions, while estimate (1) for the quantity $C_{\sigma,r}$ for the class of all uniformly almost-periodic functions belonging to $W^{(r)}$ cannot be improved, since it cannot be improved even for the class $P^{(r)}$ contained in it.

2. We formulate the main results of the note.

Theorem 1. Whatever the positive real $\sigma > 0$ and the natural number r, as $p \to \infty$ the asymptotic equality

$$C_{\sigma,r}\left(W_s^{(r)}\right) = \frac{4}{\pi^2}\ln(p+1) + O(\ln\ln\ln p), \qquad \text{where } p = \min\{\sigma, r\}. \tag{3}$$

holds.

Theorem 2. As $r \to \infty$, uniformly with respect to all $\sigma > 0$, the asymptotic equality

$$C_{\sigma,r}\left(P^{(r)}\right) = \frac{4}{\pi^2}\ln(r+1) + O(\ln\ln\ln r). \tag{4}$$

holds.

3. Let us recall some known facts and give three lemmas on which the proof of Theorem 1 is based.

The integral operator of N. I. Akhiezer—B. M. Levitan [3]

$$f_{\sigma,q}(x) = \int_{-\infty}^{\infty} f(x+u)\Psi_{\sigma,\sigma(1+1/q)}(u)\,du, \tag{5}$$

where

$$\Psi_{\sigma,\sigma(1+1/q)}(u) = \frac{q}{\pi\sigma}\,\frac{\cos\sigma u - \cos\sigma(1+1/q)u}{u^2}$$

$(q > 0)$, assigns to every continuous and bounded function $f(x)$ on $(-\infty, \infty)$ an entire function $f_{\sigma,q}(x)$ of degree $\leqslant \sigma(1+1/q)$, and moreover $f_{\sigma,q}(x) = f(x)$ if $f(x)$ is a bounded entire function on $(-\infty, \infty)$ of degree $\leqslant \sigma$.

Let $L(q)$ be the norm of the operator (5) in the space of all bounded functions on the real axis. A. F. Timan [4] established that, uniformly with respect to all $q > 0$, the asymptotic equality

$$L(q) = \frac{4}{\pi^2}\ln(q+1) + O(1) \tag{6}$$

holds.

Let the interval $I_{\sigma,q} = (\sigma, \sigma(1+1/q))$ contain n $(n \geqslant 0)$ points c_i $(c_i < c_{i+1},\ i = 1, 2, \dots, n)$; let $\varepsilon > 0$ be chosen so that the intervals $(c_i - \varepsilon, c_i + \varepsilon)$ do not intersect and belong to the interval $I_{\sigma,q}$. Consider the function, continuous and linear on the intervals $(c_i - \varepsilon, c_i)$, $(c_i, c_i + \varepsilon)$ $(i = 1, 2, \dots, n)$,

$$\varphi_{\sigma,q}(\lambda) = \varphi_{\sigma,q}(\lambda, c_1, c_2, \dots, c_n, \varepsilon) =$$

$$= \begin{cases} 1, & |\lambda| \leqslant \sigma, \\ q + 1 - \dfrac{q}{\sigma}|\lambda|, & \sigma < |\lambda| < \sigma\left(1 + \dfrac{1}{q}\right), \quad |\lambda| \notin (c_i - \varepsilon, c_i + \varepsilon), \\ 0, & |\lambda| = c_i, \\ 0, & |\lambda| \geqslant \sigma\left(1 + \dfrac{1}{q}\right). \end{cases}$$

Set

$$\widetilde{\Psi}_{\sigma,\sigma(1+1/q)}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_{\sigma,q}(\lambda) e^{-iu\lambda}\,d\lambda,$$

$$\tilde{f}_{\sigma,q}(x) = \int_{-\infty}^{\infty} f(x+u)\widetilde{\Psi}_{\sigma,\sigma(1+1/q)}(u)\,du; \tag{5'}$$

it is easy to see that $\widetilde{\Psi}_{\sigma,\sigma(1+1/q)}(u) = \Psi_{\sigma,\sigma(1+1/q)}(u)$, $\tilde{f}_{\sigma,q}(x) = f_{\sigma,q}(x)$, if the interval $I_{\sigma,q}$ is free of the points c_i.

Let $\widetilde{L}(q)$ be the norm of the operator (5′), and $\widetilde{\widetilde{L}}(q)$ the norm of the operator

$$\int_{-\infty}^{\infty} f(x+u)\left[\widetilde{\Psi}_{\sigma,\sigma(1+1/q)}(u) - \Psi_{\sigma(1-1/q),\sigma}(u)\right] du$$

in the space of all functions bounded on the real axis.

Lemma 1. If $f(x)$ is continuous and bounded on $(-\infty,\infty)$, then $\tilde{f}_{\sigma,q}(x)$ is an entire function of degree $\leq \sigma(1+1/q)$. If $f(x)$ is an entire function of degree $\leq \sigma$ bounded on $(-\infty,\infty)$, then $\tilde{f}_{\sigma,q}(x) = f(x)$.

Lemma 2. The inequalities hold

$$\tilde{L}(q) \leq \bar{L}(q) + 2n, \tag{7}$$

$$\tilde{L}(q) \leq 2(n+1). \tag{8}$$

Lemma 3. If

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x} \quad (\lambda_0 = 0,\ \lambda_{-k} = -\lambda_k,\ \lambda_k < \lambda_{k+1} \text{ for } k \geq 0,\ \lim_{k\to\infty} \lambda_k = \infty)$$

is a uniformly almost-periodic function, then

$$\tilde{f}_{\sigma,q}(x) = \sum_{|\lambda_k|<\sigma(1+1/q)} \varphi_{\sigma,q}(\lambda_k) A_k e^{i\lambda_k x}.$$

4. We shall give the proof of Theorem 1. Let I_σ be the common part of the intervals $I_{\sigma,r}$ and $(\sigma,\sigma+1)$; let $c_1, c_2, \ldots, c_n$ $(0 \leq n \leq N_f(\sigma,\sigma+1))$ be the Fourier exponents of the function $f(x)$ belonging to the interval I_σ. We shall show that for every function $f(x) \in W_s^{(r)}$

$$\max_{0\leq k\leq r} \frac{\mathrm{Sup}_x \left|f^{(k)}(x) - G_\sigma^{(k)}(\tilde{f}_{\sigma,r},x)\right|}{E_\sigma(f^{(k)})} \leq \frac{4}{\pi^2}\ln(p+1) + O(1), \tag{9}$$

where $p = \min\{\sigma, r\}$; $G_\sigma(\tilde{f}_{\sigma,r},x)$ is an entire function of degree $\leq \sigma$, realizing the best approximation to the function $\tilde{f}_{\sigma,r}(x)$.

The equalities

$$\tilde{f}_{\sigma,r}^{(k)}(x) = \int_{-\infty}^{\infty} f^{(k)}(x+u)\,\tilde{\Psi}_{\sigma,\sigma(1+1/r)}(u)\,du \quad (k=0,1,\dots,r) \tag{10}$$

are valid.

Denote by $g_\sigma(f^{(k)},x)$ an entire function of degree $\leq \sigma$, realizing the best approximation to the function $f^{(k)}(x)$. By Lemma 1,

$$g_\sigma(f^{(k)},x) = \int_{-\infty}^{\infty} g_\sigma(f^{(k)},x+u)\,\tilde{\Psi}_{\sigma,\sigma(1+1/q)}(u)\,du \quad (k=0,1,\dots,r) \tag{11}$$

for any $q > 0$.

Let $\sigma \leq r$. In consequence of (10) and Lemma 3,

$$\tilde{f}_{\sigma,r}^{(k)}(x) = \int_{-\infty}^{\infty} f^{(k)}(x+u)\,\tilde{\Psi}_{\sigma,\sigma+1}(u)\,du = S_\sigma^{(k)}(f,x), \tag{12}$$

where

$$S_\sigma(f,x) = \sum_{|\lambda_k|\leq\sigma} A_k e^{i\lambda_k x},$$

and therefore

$$G_\sigma^{(k)}(\tilde{f}_{\sigma,r},x) = \tilde{f}_{\sigma,r}^{(k)}(x) \quad (k=0,1,\dots,r). \tag{13}$$

Putting $q=\sigma$ in equality (11), we obtain from (11), (12), and (13)

$$\operatorname{Sup}\left|f^{(k)}(x) - G_\sigma^{(k)}(\tilde{f}_{\sigma,r},x)\right| \leq \{\tilde{L}(\sigma)+1\}E_\sigma(f^{(k)}) \quad (k=0,1,\dots,r).$$

From the last inequality, by virtue of (7), it follows that for $\sigma \leqslant r$

$$\max_{0\leqslant k\leqslant r} \frac{\operatorname{Sup}_x \left|f^{(k)}(x) - G_\sigma^{(k)}(\tilde{f}_{\sigma,r},x)\right|}{E_\sigma(f^{(k)})} \leqslant L(\sigma) + 2N_f(\sigma,\sigma+1) + 1. \tag{14}$$

Let $\sigma > r$. It is easy to see that

$$\int_{-\infty}^{\infty} [f(x+u) - g_\sigma(f,x+u)][\tilde{\Psi}_{\sigma,\sigma(1+1/r)}(u) - \Psi_{\sigma(1-1/r),\sigma}(u)]\,du =$$

$$= \tilde{f}_{\sigma,r}(x) - \Phi_\sigma(x),$$

where $\Phi_\sigma(x)$ is some entire function of degree $\leqslant \sigma$; therefore, from inequality (8) one obtains the estimate

$$E_\sigma(\tilde{f}_{\sigma,r}) \leqslant 2[1 + N_f(\sigma, \sigma+1)]E_\sigma(f). \tag{15}$$

For any function $f(x) \in W^{(r)}$ (see ([1],[3])),

$$E_\sigma(f) \leqslant \frac{\pi}{2\sigma^k} E_\sigma(f^{(k)}) \qquad (k = 0, 1, \dots, r). \tag{16}$$

From S. N. Bernstein' s inequality ([3]) and inequalities (15) and (16) it follows that

$$\operatorname{Sup}_x \left|\tilde{f}^{(k)}_{\sigma,r}(x) - G^{(k)}_\sigma(\tilde{f}_{\sigma,r}, x)\right| \leqslant$$

$$\leqslant \pi \left(1 + \frac{1}{r}\right)^k [1 + N_f(\sigma, \sigma+1)]E_\sigma(f^{(k)}) \qquad (k = 0, 1, \dots, r).$$

Taking $q = r$ in equality (11), we obtain from (10), (11), and (7)

$$\operatorname{Sup}_{x'} \left|\tilde{f}^{(k)}(x) - \tilde{f}^{(k)}_{\sigma,r}(x)\right| \leqslant [L(r) + 2N_f(\sigma, \sigma+1) + 1]E_\sigma(f^{(k)})$$

$$(k = 0, 1, \dots, r).$$

From the last two inequalities it follows that, for $\sigma > r$,

$$\max_{0\leqslant k\leqslant r} \frac{\operatorname{Sup}_x \left|f^{(k)}(x) - G^{(k)}_\sigma(\tilde{f}_{\sigma,r}, x)\right|}{E_\sigma(f^{(k)})} \leqslant L(r) + (2+\pi e)N_f(\sigma, \sigma+1) + \pi e + 1. \tag{17}$$

Estimate (9) is a consequence of inequalities (14), (17), and the asymptotic equality (6). From (9) we obtain the inequality

$$C_{\sigma,r}(W^{(r)}_s) \leqslant \frac{4}{\pi^2}\ln(p+1) + O(1), \qquad \text{where } p = \min\{\sigma, r\},$$

which, by virtue of (2), leads to the asymptotic equality (3).

5. We do not present here the proof of Theorem 2, based on the effective construction of a function $g_\sigma(x) \in P^{(r)}$ for which

$$C_{\sigma,r}(g_\sigma) \geqslant \frac{4}{\pi^2}\ln(r+1) + O(\ln\ln\ln r).$$

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Received
9 II 1962

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.