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Abstract

Full Text

Mathematics

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Structure of the Set of Solutions of Equations of Parabolic Type

(Presented by Academician I. G. Petrovskii, March 28, 1962)

The main result of the present article can be formulated quite simply: if a mixed boundary-value problem for a parabolic equation has two solutions, then it has a continuum of solutions. For ordinary differential equations an analogous assertion is contained in the well-known theorem of Kneser–Hukuhara.

1. A completely continuous operator B, acting in a Banach space E, will be called *smoothable* on a set $\mathfrak{M} \subset E$ if, for every $\varepsilon > 0$, one can construct a completely continuous operator B_ε such that $\|B_\varepsilon x - Bx\| < \varepsilon$ $(x \in \mathfrak{M})$, and the equation $x = B_\varepsilon x + y$ cannot have on $\mathfrak{M}$ more than one solution for sufficiently small (in norm) y. The assertions given below are consequences of the following *principle of connectedness*, proved in ([2]).

Theorem 1 ([2]). *Suppose that the rotation γ of the completely continuous vector field $x - Bx$ on the boundary Γ of a bounded domain G is nonzero. Suppose that the operator B is smoothable on $G + \Gamma$.*

Then the set of fixed points of the operator B lying in G is connected.

If E is finite-dimensional, then the rotation γ under the conditions of Theorem 1 is obviously equal to $+1$ or -1. The question remains open whether this fact is true in the case of infinite-dimensional E.

2. Let us first consider the equation

$$\frac{dv}{dt} + Av = f(t, v) \qquad (0 < t \leq t_0) \tag{1}$$

with the initial condition

$$v(0) = v_0 \tag{2}$$

in a Hilbert space H. Here and below dv/dt denotes the strong limit of the corresponding difference quotient.

Assume that A is a positive definite self-adjoint operator having a completely continuous inverse. Let $f(t, v)$ be a continuous nonlinear operator acting from

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$[0, t_0] \times H$ into the space $C[0, t_0]$ of functions continuous on $[0, t_0]$ with values in H and bounded on every bounded set.

A solution of problem (1)–(2) is called *classical* if it is continuous for $t \geq 0$ and continuously differentiable for $t > 0$. Every classical solution is at the same time a continuous solution of the integral equation

$$v(t) = e^{-tA} v_0 + \int_0^t e^{-(t-s)A} f[s, v(s)]\, ds. \tag{3}$$

Continuous solutions of equation (3) will be called *generalized* solutions of problem (1)–(2). It is known ([3,4]) that every generalized solution is classical if $f(t, v)$ satisfies some Hölder condition

$$\|f(t_1, v_1) - f(t_2, v_2)\| \leq L(R)(|t_1 - t_2|^\mu + \|v_1 - v_2\|^\nu)$$

$$(\|v_1\|, \|v_2\| \leq R; \quad 0 \leq t_1, t_2 \leq t_0). \tag{4}$$

Theorem 2. Let, for any ε and R, one be able to construct an operator $f_\varepsilon(t, v)$ satisfying the conditions

$$\|f_\varepsilon(t, v) - f(t, v)\| \leq \varepsilon \quad (0 \leq t \leq t_0,\ \|v\| \leq R), \tag{5}$$

$$\|f_\varepsilon(t, v_1) - f_\varepsilon(t, v_2)\| \leq L(\varepsilon; R)\|v_1 - v_2\| \quad (\|v_1\|, \|v_2\| \leq R). \tag{6}$$

Then the generalized solutions of problem (1)–(2), considered on some interval $[0, t^*] \subset [0, t_0]$, form a connected set in $C[0, t^*]$.

If condition (4) is additionally satisfied, then the set of classical solutions is connected.

3. If H is finite-dimensional, then $f_\varepsilon(t, v)$ can always be constructed. In the case of infinite-dimensional H we have to impose an additional restriction on the initial condition.

Theorem 3. Let $v_0 \in D(A^\beta)$, where $\beta > 0$. Then the assertion of Theorem 2 is valid.

For applications, cases are important in which $f(t, v)$ does not possess the continuity property (for differential equations this means that the nonlinear terms contain derivatives and essential nonlinearities).

Theorem 4. Let the operator $f(t, A^{-\alpha} v)$, for some $\alpha \in (0, 1)$, be continuous jointly in its variables and bounded on every bounded set. Let $v_0 \in D(A^\beta)$, where $\beta > \alpha$. Then the assertion of Theorem 2 is valid.

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The generalized solutions under the conditions of Theorem 4 will be classical if (see ([3],[4])), for example, the weakened Hölder condition is satisfied

$$\|f(t_1, A^{-\alpha}w_1) - f(t_2, A^{-\alpha}w_2)\| \le L(R)\big(|t_1 - t_2|^{\mu} + \|w_1 - w_2\|^{\nu}\big).$$

4. Let us now consider problem (1)–(2) in a Banach space. Suppose that $-A$ is the generating operator of a strongly continuous semigroup (see ([7])), which we shall denote by e^{-tA}. Then problem (1)–(2) can be reduced to the study of the integral equation (3). For the study of this equation, as it turns out, it is sufficient that fractional powers of the operator A be defined, that the complete continuity of the operator A^{-1} imply the complete continuity of the operators $A^{-\alpha}$ for $\alpha > 0$, and that the operators $A^{\alpha}e^{-tA}$, for $\alpha, t > 0$, be continuous and have norms not exceeding $L(\alpha)t^{-\alpha}$. The requirements listed are fulfilled (see ([5],[6],[8],[9])) if

$$\|(A + \lambda I)^{-1}\| \le C(1 + |\lambda|)^{-1} \quad (\operatorname{Re}\lambda \ge 0); \tag{7}$$

such operators we shall call **strongly positive**.

Theorem 5. If the operator A is strongly positive and A^{-1} is completely continuous, then Theorems 2, 3, and 4 are true for equations in a Banach space.

5. Let us pass to an equation more complicated than (1),

$$\frac{dv}{dt} + A(t)v = f(t, v) \tag{8}$$

with a variable operator $A(t)$. We shall assume that $A(t)$, for fixed t, is strongly positive, and moreover that the constant C in condition (7) does not depend on t. We shall suppose that the operators $A(t)$ have a common domain of definition and that for $0 \le t, \tau, s \le t_0$

$$\|[A(t) - A(\tau)]A^{-1}(s)\| \le C|t - \tau|^{\delta}, \tag{9}$$

where δ is some number in $(0, 1)$.

Under the assumptions listed, the homogeneous problem $dv/dt + A(t)v = 0$ $(s < t \le t_0)$, $v(s) = v_0$, as shown in ([4]), has a classical solution $v(t) = U(t, s)v_0$, where the operator $U(t, s)$ has many properties

the semigroup $e^{-(t-s)A}$. Problem (8)–(2) can be reduced to the equation

$$v(t) = U(t, 0)v_0 + \int_0^t U(t, s)f[s, v(s)]\,ds \tag{10}$$

or to the equation

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$$w(t) = A_0^{\beta} U(t,0) A_0^{-\beta} w_0 + \int_0^t [A^{\beta} U(t,s) A_0^{\gamma}] A_0^{-\gamma} f[s, A_0^{-\beta} w(s)]\, ds, \tag{11}$$

where $A_0 = A(0)$, $0 \leqslant \beta < 1$, $0 < \gamma < \min(\delta, 1-\beta)$. An investigation of these equations shows that Theorem 5 carries over without change to problem (8)–(2).

We note that the results of the paper ([10]) make it possible to generalize Theorem 5 also to the case when the common domain of definition has only some fractional power $A^{\rho}(t)$ of the operator $A(t)$.

6. Let us now consider the quasilinear problem

$$\frac{dv}{dt} + A(t,v)v = f(t,v), \qquad v(0) = v_0. \tag{12}$$

We shall assume that: a) the operator $A = A(0, v_0)$ is strongly positive and its inverse A^{-1} is completely continuous; b) the operators $A(t, A^{-\alpha} w)$, where α is some number in $(0,1)$, are strongly positive and have a common domain of definition, and moreover

$$\left\|[A(t, A^{-\alpha} w) - \lambda I]^{-1}\right\| \leq C(R)(1+|\lambda|)^{-1} \qquad (\operatorname{Re}\lambda \geqslant 0,\ \|w\| \leqslant R) \tag{13}$$

and, for $0 \leqslant t_1, t_2 \leqslant t_0$, $\|w_1\|, \|w_2\| \leqslant R$,

$$\left\|[A(t_1, A^{-\alpha} w_1) - A(t_2, A^{-\alpha} w_2)] A^{-1}\right\| \leq C(R)\big(|t_1 - t_2|^{\mu} + \|w_1 - w_2\|^{\nu}\big); \tag{14}$$

c) the operator $f(t, A^{-\alpha} w)$ is continuous in the aggregate of the variables; d) $v_0 \in D(A^{\gamma})$, where $\gamma > \alpha$. Under these assumptions, in ([4]) a local existence theorem is proved for a generalized solution of problem (12).

By $C_{\eta}[0, t^*]$ we shall denote the space of functions defined on $[0, t^*]$, with values in E, satisfying a Hölder condition with exponent η.

Theorem 6. Suppose that conditions a)–d) are fulfilled. Suppose that there is a basis in E. Then the solutions of problem (12) satisfying on some interval $[0, t^*]$ a Hölder condition with some exponent $\eta < \gamma - \alpha$ form a connected set in $C_{\eta}[0, t^*]$.

The results of the paper ([10]) make it possible to generalize this theorem to the case when the operators $A^{\rho}(t, v)$ have a common domain of definition for some ρ.

7. In Theorems 2–6 the connectedness is asserted of the set of solutions defined on some small, generally speaking, interval. If for the solutions of the corresponding problems there are a priori estimates on the whole

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interval under consideration $[0, t_0]$, then in Theorems 2–6 one can speak of the connectedness of the set of solutions in spaces of functions defined on the whole interval $[0, t_0]$.

8. Applications, for example, to mixed problems for parabolic equations are obtained by the standard scheme (see ([3,4])). We give examples.

Let Ω be a bounded domain of n-dimensional space with twice continuously differentiable boundary S. Consider the equation

$$\frac{\partial v}{\partial t} = \sum a_{ij}\left(t, x, v, \frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right)\frac{\partial^2 v}{\partial x_i \partial x_j} + f\left(t, x, v, \frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right). \quad (15)$$

Suppose that the functions a_{ij}, $\dfrac{\partial}{\partial x_k} a_{ij}$, f satisfy, in the aggregate of all variables, some Hölder condition and the condition

$$\sum a_{ij}\xi_i\xi_j \geqslant \lambda \sum \xi_i^2,$$

where λ and the constants in the Hölder conditions may depend on R, where R is the bound of the absolute values of the values under consideration $v, \partial v/\partial x_1, \dots, \partial v/\partial x_n$.

Theorem 7. *The set of solutions of equation (15) satisfying the conditions*

$$v(t, x) = 0 \quad (0 < t \le t_0;\ x \in S), \qquad v(0, x) = v_0(x) \quad (x \in \Omega + S) \qquad (16)$$

is connected, if $v(0, x) \in W_p^2(\Omega)$ for some $p > n$.

Theorem 7 remains valid if, instead of the first boundary condition, one considers the second:

$$\sum a_{ij}\frac{\partial v}{\partial x_j}\cos(\mathbf{N}_x, \mathbf{x}_i) + \sigma v = 0 \qquad (0 < t \le t_0,\ x \in S)$$

(here $\mathbf{N}_x$ is the normal to S at the point x). In this case, in the corresponding problem (12), the operator $A(t, v)$ has (see (10)) a variable domain of definition.

If the coefficients a_{ij} of equation (15) do not depend on the derivatives and if the right-hand side satisfies the inequalities

$$|f(t, x, v; 0, \dots, 0)| \le k_1 v^2 + k_2,$$

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$$|f(t,x,v;v_1,\dots,v_n)| \le k_3(R) + k_4(R)\sum |v_i|^{2-\varepsilon} \qquad (|v| \le R),$$

where $\varepsilon > 0$, then a priori estimates are valid for the solutions of problem (15)–(16) (see (12)). Therefore the set of solutions considered on any fixed finite interval of variation of t is connected. In an analogous way one may use a priori estimates of solutions established in other works (see, for example, (13, 14)).

As a second example, let us consider the equation

$$\frac{\partial v}{\partial t} + L(t,x,v)v = f(t,x,v), \tag{17}$$

where $L(t,x,v)$ is an elliptic operator of order $2m$ or a system strongly elliptic in the sense of M. I. Vishik. The coefficients of the operator $L(t,x,v)$ and the right-hand side may contain derivatives of v with respect to the spatial variables up to order $2m-1$. From the results obtained in (4), and from Theorem 6, it follows that for equation (17) the theorem on the connectedness of the set of solutions is valid.

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