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Abstract

Full Text

MATHEMATICAL PHYSICS

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ON ONE VARIANT OF THE STEFAN PROBLEM*

(Presented by Academician S. L. Sobolev, 16 X 1961)

A one-phase problem is considered which is a generalization of the classical Stefan problem; namely, one seeks $y(t)$ and $u(x, t)$ such that u and $\partial u / \partial x$ are defined and continuous in the closure of the parabolic domain $D : \{0 < x < y(t); t > 0\}$, while u satisfies inside D the equation

$$a^2 \frac{\partial^2 u}{\partial x^2} + F \left(x, t, u, \frac{\partial u}{\partial x}, y, \dot{y} \right) = \frac{\partial u}{\partial t} \quad (a = \text{const}) \quad (1)$$

and on the boundary of D the conditions

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = f(t, u)|_{x=0}; \quad u(x, 0) = \varphi(x); \quad (2)$$

$$u(y(t), t) = \psi[y(t)]; \quad \left. \frac{dy}{dt} = \Phi \left(t, u, \frac{\partial u}{\partial x}, y \right) \right|_{x=y}; \quad y(0) = l > 0.$$

It is assumed that F , f , and Φ have bounded partial derivatives with respect to all their arguments, and that ψ and φ have bounded derivatives respectively up to the second and up to the third order inclusive. Under these assumptions the following theorems are proved:

Theorem 1. Put

$$w(t) = u(0, t); \quad q(x, t) = \frac{\partial}{\partial x} u(x, t); \quad v(t) = \frac{\partial}{\partial x} u(y(t), t); \quad \dot{y}(t) = z(t). \quad (3)$$

Then, if u, y are a solution of problem (1), (2), then u, w, q, v, y, z are solutions of the system of integral equations

$$\begin{aligned}
 u = & -a^2 \int_0^t f(\tau, w)G(x, 0, t - \tau) d\tau + \int_0^l \varphi(\xi)G(x, \xi, t) d\xi \\
 & + \int_0^t d\tau \int_0^{y(\tau)} F(\xi, \tau, \dots)G(x, \xi, t - \tau) d\xi \\
 & + a^2 \int_0^t [v(\tau) + \psi(y(\tau))z(\tau)]G(x, y(\tau), t - \tau) d\tau \\
 & - a^2 \int_0^t \psi(y(\tau)) \frac{\partial}{\partial \xi} G(x, y(\tau), t - \tau) d\tau \equiv U(t, x | u, w, q, v, y, z);
 \end{aligned} \tag{4}$$

$$w = U|_{x=0};$$

$$\begin{aligned}
 q(x, t) = & a^2 \int_0^t f(\tau, w) \frac{\partial}{\partial \xi} g(x, 0, t - \tau) d\tau + \int_0^l \varphi(\xi)g(x, \xi, t) d\xi \\
 & - \int_0^t d\tau \int_0^{y(\tau)} F(\xi, \tau, \dots) \frac{\partial}{\partial \xi} g(x, \xi, t - \tau) d\xi \\
 & - a^2 \int_0^t v(\tau) \frac{\partial}{\partial \xi} g(x, y(\tau), t - \tau) d\tau +
 \end{aligned}$$

* Reported at the Fourth All-Union Mathematical Congress.

$$+ \int_0^t \psi(y(\tau))z(\tau)g(x, y(\tau), t - \tau) d\tau \equiv Q(t, x | u, w, q, v, y, z);$$

$$v(t) = 2Q|_{x=y(t)}, \quad y(t) = l + \int_0^t z(\tau) d\tau; \quad z(t) = \Phi(t, \psi(y), v, y).$$

Here g and G are the Green's functions of the first and second boundary-value problems for the heat equation on the half-line $x > 0$, i.e.

$$g(x, \xi, t) = E(x - \xi, a^2t) - E(x + \xi, a^2t);$$

$$G(x, \xi, t) = E(x - \xi, a^2t) + E(x + \xi, a^2t); \quad E(x, t) = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}}.$$

Theorem 2. If u, w, q, v, z, y are solutions of system (4) satisfying a Lipschitz condition in all arguments, then u, y are solutions of the original problem (1), (2).

Theorem 3. The unique solution of system (4) satisfying the conditions of Theorem 2 can be constructed by Picard's method of successive approximations, if the process is begun with arbitrary functions $u_0, w_0, v_0, q_0, y_0, z_0$ having bounded partial derivatives in all their arguments and satisfying the conditions

$$\dot{\varphi}(0) = f(0, w_0); \quad \dot{\varphi}(l) = q_0(l, 0) - v_0(0); \quad \dot{y}_0(t) = z_0(t). \quad (5)$$

The iterative process converges on some small time interval $(0, T)$, where T depends on the estimates of $f, F, \Phi, \psi, \varphi$, on the estimates of all the above-mentioned partial derivatives of f, \dots, φ , and on $l = y(0) > 0$, with $T \rightarrow 0$ as $l \rightarrow 0$.

Theorem 3*. A solution of system (4) can be constructed for sufficiently small $t > 0$ by the finite-difference method.

The computational scheme in this case is analogous to that which was used earlier [1].

Theorem 4. The solution of system (4) is stable in the following sense: suppose that, along with $a^2, l, F, f, \varphi, \psi, \Phi$, there are given $a^{2*}, l^*, F^*, f^*, \varphi^*, \psi^*, \Phi^*$ possessing the required differentiability properties. Let $u^*, q^*, w^*, v^*, y^*, z^*$ be the solution of the system obtained from (4) by replacing a^2, l, \dots, Φ by $a^{2*}, l^*, \dots, \Phi^*$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, on some small interval $(0, T)$ independent of ε , the inequalities

$$|u(x, t) - u^*(x, t)| < \varepsilon; \quad |q(x, t) - q^*(x, t)| < \varepsilon$$

$$\text{for } 0 \leq x \leq \min(y(t), y^*(t));$$

$$|v(t) - v^*(t)| < \varepsilon; \quad |y(t) - y^*(t)| < \varepsilon; \quad |z(t) - z^*(t)| < \varepsilon \quad (6)$$

hold as soon as the moduli of the differences a^2 and a^{2*} ; l and l^* ; F, f, Φ and respectively F^*, f^*, Φ^* and all their partial derivatives of first order; ψ, ψ^* and their derivatives up to second order; φ, φ^* and their derivatives up to third order inclusive do not exceed δ .

The results remain valid for the analogous problem with any finite number of phase-separation boundaries, provided that all phases coexist at the initial moment.

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REFERENCES

1. L. I. Rubinshtein, *Izv. Vyssh. uchebn. zaved.*, Mathematics, No. 4 (1958).

Note: Figure translations are in progress. See original paper for figures.

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