V. P. PALAMODOV

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Abstract

Full Text

V. P. PALAMODOV

THE GENERAL FORM OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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As is known, every solution of an ordinary linear equation with constant coefficients and with zero right-hand side is a linear combination of the corresponding exponentials (exponential polynomials). For partial differential equations with constant coefficients

$$p(D)u = 0, \qquad D = i\frac{\partial}{\partial x_1}, \dots, i\frac{\partial}{\partial x_n}. \tag{1}$$

this fact was generalized by L. Ehrenpreis in ([1]) and independently by the author in ([4]) as follows: every solution of equation (1) is the Fourier transform of a measure concentrated on the complex variety of roots of the polynomial $p(s)$.*

In ([4]) a formulation was given and a proof of this theorem was outlined for the cases: 1) the solution u is an entire function; 2) the solution u is an ordinary or generalized function growing at infinity no faster than $\exp(A|x|^\alpha)$ for certain $A > 0$ and $\alpha > 0$. In the present note we indicate the general form of the solutions of (1) in the broadest spaces $S_0^{\beta'}$ and $\mathcal{E}_0^\beta$ (see ([2,4])) of ordinary and generalized functions.

We begin with the definition of the functional spaces needed below. Let $K \subset R^n$ be some compact set. By $S_0^\beta(K)$, $1 < \beta \le \infty$, we denote the space of all finite functions whose supports belong to K, infinitely differentiable for $\beta = \infty$, and belonging to the Gevrey class of order β for $1 < \beta < \infty$, with the natural topology. Let $\Omega \subset R^n$ be some open set and let $K_1 \subset K_2 \subset \cdots$ be an increasing sequence of compact subsets of Ω tending to Ω. Put $S_0^\beta(\Omega) = \lim\operatorname{ind} S_0^\beta(K_i)$. By $\mathcal{E}_0^\beta(\Omega)$ we denote the space of all functions in the domain Ω that are infinitely differentiable for $\beta = \infty$ and belong to the Gevrey class of order β for $1 < \beta < \infty$.

Let $p(s)$ be the polynomial corresponding to the operator in (1). Construct a factorization $p(s) = p_1(s) \cdots p_r(s)$ of this polynomial such that each polynomial $p_i(s)$ has no multiple factors and is divisible by $p_{i+1}(s)$. Such a factorization is unique. Let N_i be the complex variety of roots of the polynomial $p_i(s)$, and let

N_i^0 be the difference between the set N_i and a 1-neighborhood of the set of its singular points; $n_i(s)$ is the normal vector to N_i^0 at the point s.

By s_a we denote the open ball of radius a with center at the origin. We formulate the main theorem.

Theorem 1. Each generalized function $u \in S_0^{\beta'}(s_{na})$ (infinitely differentiable function $u \in \mathcal{E}_0^\beta(s_{na})$) that is a solution of equation (1) admits the representation

$$u(x) = \sum_{i \le r} \int_{N_i^0} e^{ixs} \frac{\partial^i}{\partial n_i^i(s)} \, d\mu_i(s), \tag{2}$$

* The theorem announced by L. Ehrenpreis in ([1]) also covers systems of equations of the form (1); however, in the general case this theorem is false.

in which the integral converges and coincides with u as a functional belonging to $S_0^{\beta'}(s_a)$ ($\mathcal{E}_0^\beta(s_a)$). In addition, the measure

$$\exp(a|\tau|_{(+)} b|\sigma|^{1/\beta}) \, d\mu_i(s),$$

where for $\beta = \infty$ we put $|\sigma|^{1/\beta} = \log(|\sigma| + 1)$, has bounded variation for every i and some b (any b).

Thus, if u is a solution of (1) in some ball, then in the concentric ball of radius n times smaller the functional u admits the representation (2). Hence it follows:

Theorem 1a. Let $\Omega \subset R^n$ be an open set. In order that a function $u \in S_0^{\beta'}(\Omega)$ ($u \in \mathcal{E}_0^\beta(\Omega)$) satisfy equation (1), it is necessary and sufficient that for every point $x_0 \in \Omega$ there be a number $a > 0$ such that the functional $u(x - x_0)$ admits the representation (2).

In the case $\Omega = R^n$ this result can be formulated more simply. We note that $S_0^\beta(R^n)$ and $\mathcal{E}_0^\beta(R^n)$ coincide with the spaces S_0^β and $\mathcal{E}_0^\beta$, defined in ([2,4]).

Theorem 1b. The general form of the solutions of equation (1) in the space $S_0^{\beta'}$ ($\mathcal{E}_0^\beta$) is given by equality (2), where the integral converges and coincides with u as a functional belonging to $S_0^{\beta'}$ ($\mathcal{E}_0^\beta$), and the measure

$$\exp(a|\tau|_{(+)} b|\sigma|^{1/\beta}) \, d\mu_i(s)$$

has bounded variation for all i, a, and some b (any b).

The representation (2) turns out to be convenient for solving a number of problems; in particular, from Theorems 1, 1a, 1b, the results ([4]), and the duality formulas ([2,3]), all the results of the theory of hypoelliptic operators can be obtained. Here we note only one consequence of the formulated theorems and the results ([4]), which gives the solution of a known problem posed by Hadamard.

Corollary. Let Φ be one of the spaces $S_\alpha^{\beta'}$, $\mathcal{E}_\alpha^\beta$, $\alpha < 1$ (see ([2,4])).

1) Suppose that $p(s) = q(s)r(s)$, where the polynomials q and r are relatively prime. Then every solution of equation (1) belonging to Φ is the sum of solutions of the equations

$$q(D)w = 0, \qquad r(D)v = 0,$$

belonging to Φ.

2) Suppose that $p(s) = q^k(s)$, $k > 0$, and that the polynomial q has no multiple factors. Then every solution of equation (1) belonging to Φ is represented in the form

$$u = \sum_{|\nu| \le k-1} x^\nu u_i,$$

where all $u_i \in \Phi$ satisfy the equation $q(D)u_i = 0$.

We shall outline the proof of Theorem 1 for the case $u \in S_0^\infty(s_{na})$. For simplicity suppose that the polynomial $p(s)$ has no multiple factors. Let N_ν denote the complex algebraic variety of roots of the polynomial $p(s)$ of multiplicity not less than ν, i.e. the set of solutions of the system $D^i p(s) = 0$, $|i| \le \nu - 1$. These varieties form a decreasing sequence and

$$N_1 = \overline{N}_1, \quad C^n = N_0 \supset N_1 \supset \cdots \supset N_m \supset N_{m+1} = \Lambda,$$

where m is the degree of the polynomial $p(s)$. Obviously

$$C^n = \bigcup N_\nu \setminus N_{\nu+1}.$$

Let $\theta(s)$ denote the function defined in C^n which on the set $N_\nu \setminus N_{\nu+1}$ is equal to

$$\rho(s, N_{\nu+1}/(|s|^2 + 1)).$$

Lemma 1. There exists a countable set of differential operators $\mathcal{D}_\alpha(s, D)$ with polynomial coefficients on each set $N_\nu \setminus N_{\nu+1}$ and of functions $\chi_{i\alpha}(s)$, satisfying the inequalities $|\chi_{i\alpha}(s)| \le$

$\le \theta^{-q_i,\alpha}(s)$, such that for any entire function $\varphi(s)$ and any point $s \in N_1$, the series

$$\varphi_s(s + \xi) = \sum \varkappa_{i\alpha}(s)\, \xi^i \mathcal{D}_\alpha(s, D)\varphi(s)$$

converges absolutely in some neighborhood U_s of the form $U_s = \{s' : |s' - s| \le c\theta^q(s)\}$. Moreover, $\varphi - \varphi_s = p\psi$, where ψ is a function analytic in U_s,

$$\max_{|s'-s| \le \frac{c}{2}\theta^q(s)} |\varphi_s(s')| \le C \max_{s' \in U_s \cap N_1} |\varphi(s')|$$

and, with certain measures $\mu_s^\alpha(s')$,

$$\mathcal{D}_\alpha(s, D)\varphi(s) = \int_{U_s \cap N_1} \varphi(s')\, d\mu_s^\alpha(s').$$

The idea of the proof of Lemma 1 is as follows. Let s_0 be a point belonging to N_1; let (ξ, η) be a splitting of the variables s into two groups, with ξ a single variable and $s_0 = (\xi_0, \eta_0)$. Further, let $\xi_i(\eta)$, $i = 1, \dots, k$, be the roots of the polynomial $p(s)$ close to ξ_0 when η is close to η_0. Then the function $\varphi_{s_0}(s)$, for each fixed η, is a polynomial in ξ of degree $k-1$, coinciding with $\varphi(s)$ at the points $\xi_i(\eta)$, $i = 1, \dots, k$.

Remark. The system of operators $\mathcal{D}_\alpha(s, D)$ constructed in this lemma has the following property at each point s: if $\psi[\xi]$ is a formal power series, then, in order that $\psi[\xi]$ belong to the ideal generated by the series

$$\sum \xi^i \frac{D^i}{i!} p(s)$$

in the space of all power series, it is necessary and sufficient that $\mathcal{D}_\alpha(s, \delta)\psi[\xi] = 0$ for every α. This system of operators coincides with the system of operators, bearing the same notation, constructed in (5).

The space dual to $S_0^\infty(s_a)$ will be denoted by Z_a. By the Paley–Wiener theorem, Z_a is the space of entire functions of first order, rapidly decreasing along the real subspace, with topology determined by the countable set of norms

$$\|\varphi\|^{a,j} = \sup_{s \in C^n} e^{-a|\tau|}\, |s^j \varphi(s)|. \tag{3}$$

By C_a we denote the space of all continuous functions in C^n for which all the norms (3) are defined.

Definition. The completion of the space Z_a in the topology determined by the countable set of norms

$$\|\varphi\|_{N_1}^{a,j} = \sup_{s \in N_1} e^{-a|\tau|}\, |s^j \varphi(s)|, \tag{4}$$

will be denoted by $Z_a(N_1)$. In view of Lemma 1, every element $f \in Z_a(N_1)$ is a continuous function given on the set N_1, having the property that, for each point $s \in N_1$, the series

$$f_s(s+\xi) = \sum \varkappa_{i\alpha}(s)\, \xi^i \int_{U_s \cap N_1} f(s')\, d\mu_s^\alpha(s')$$

converges absolutely in U_s and coincides with f in $U_s \cap N_1$.

In the following lemma we shall give a description of the complex measure that determines the zero functional on the space Z_a.

Lemma 2. Every measure μ, given in C^n, determining a continuous functional on C_{pa} and equal to zero on the subspace $Z_{a+\varepsilon}$, for any $\varepsilon > 0$, admits the representation

$$(\mu, \varphi) = \sum_{|j|\le k} \int_{|x|\le a} e^{-2a|\tau|} \int_{\Gamma_\tau} e^{itxs} s^j \varphi(s)\, ds\, d\gamma(x,\tau) +$$

$$+ \sum_{|j|\le k} \int_{|x|>a} e^{-a|\tau|} \int_{\gamma_\tau} e^{ixs} s^j \varphi(s)\, ds\, d\lambda(x,\tau), \tag{5}$$

where the right-hand side converges and coincides with μ as a functional belonging to C'_a; $d\gamma$ and $d\lambda$ are complex measures of bounded variation; T_τ is a contour consisting of the manifold $R^n + +i\tau$; Γ_τ is a contour consisting of the manifold $R^n - +i\tau$ with positive orientation and the manifold R^n with negative orientation.

Lemma 2 follows from the easily verified inequality

$$\|\varphi - \varphi * \chi_a\|^{na,j} \le C_j \|\varphi\|_0^{a,j},$$

where $\chi_a(x)$ is a certain infinitely differentiable function equal to zero for $|x| \ge a$ and to one for $|x| \le a - \varepsilon$, and

$$\|\varphi\|_0^{a,j} =$$

$$= \max\left\{ \sup_{\tau,\, |x|\le a} e^{-2a|\tau|} \left| \int_{\Gamma_\tau} e^{ixs} s^j \varphi(s)\, ds \right|, \sup_{\tau,\, |x|>a} e^{-a|\tau|} \left| \int_{\Gamma_\tau} e^{ixs} s^j \varphi(s)\, ds \right| \right\}.$$

The proof of Theorem 1 is completed as follows. Let the functional $u \in S_a^\infty(s_{na})$ be a solution of equation (1). Then the functional $\tilde{u}$ belongs to Z'_{na} and is a solution of the equation $p(s)\tilde{u} = 0$. Extend this functional to the space C_{na}, represent it in the form of a measure by means of the Riesz theorem, and multiply this measure by $p(s)$. The resulting measure μ belongs to C'_{na}, gives the zero functional on Z_{na}, and, consequently, admits the representation (5). Substitute this representation into the scalar product $(\mu, \varphi/p)$, where $\varphi \in Z_a$, and deform the contours Γ_τ and T_τ in (5) near N_1 so that they do not intersect N_1. Then contract these contours into a neighborhood of the set N_1, which is the union of all the domains U_s. In this way we have extended the functional μ to the space $Z_a(N_1)$. But, by construction, $(\tilde{u}, \varphi) = (\mu, \varphi/p)$, whence also $\tilde{u} \in Z'_a(N_1)$. The space $Z_a(N_1)$ is complete in the topology determined by the norms (4); therefore, applying the Riesz theorem once again, we obtain the representation (2).

Remark. Denote

$$Z(Z(N_1)) = \lim_{a\to\infty} \operatorname{ind} Z_a(Z_a(N_1)).$$

It follows from the arguments given above that the natural embedding Z into $Z(N_1)$ determines a topological isomorphism

$$Z/pZ \simeq Z(N_1),$$

where pZ is the ideal generated by the polynomial $p(s)$ in Z. An analogous assertion is valid for any basic space of entire functions.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.