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Abstract

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THE CAUCHY PROBLEM FOR SECOND-ORDER PARABOLIC EQUATIONS WITH GROWING COEFFICIENTS

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In this paper we consider the Cauchy problem for a second-order parabolic equation with real coefficients:

$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{N}a_{ij}(x,t)D_{iD}ju+\sum_{k=1}^{N}b_k(x,t)D_ku+c(x,t)u; \tag{1}$$

$$u\big|_{t=0}=u_0(x) \tag{2}$$

$$(0\leqslant t\leqslant T\leqslant\infty;\quad x=(x_1,\dots,x_N)\in E_N;\quad D_i=\frac{\partial}{\partial x_i};\quad \sum a_{ij}(x,t)\xi_i\xi_j\geqslant$$

$$\geqslant a(x,t)\sum\xi_k^2;\quad a(x,t)>0).$$

The Cauchy problem for parabolic equations (and systems) has been studied by a number of authors under various restrictions on the growth of the coefficients as $|x|\to\infty$. The most general results were obtained by Ya. I. Zhitomirskii (^{1}) and S. D. Eidelman (^{2}) by constructing a fundamental solution. As applied to equation (1), these results are formulated as follows: if $a(x,t)\geqslant a_0>0$, $a_{ij}=O(1)$, $b_k=O(|x|)$, $c=(|x|^2)$, $D_ia_{ij}=O(|x|)$, $D_{iD}ja_{ij}=O(|x|^2)$, $D_kb_k=O(|x|^2)$, then the solution of the Cauchy problem (1)–(2) exists and is unique in the class of functions satisfying the condition $|u(x,t)|\leqslant\text{const}\cdot e^{k|x|^2}$, $k>0$.

We shall give another approach to the Cauchy problem, based on general operator considerations analogous to those used in a number of papers in the study of the mixed problem for equation (1) (^{3,4}). This approach makes it possible, without constructing a cumbersome analytic apparatus, to investigate the classes of uniqueness and existence of solutions of the Cauchy problem, and the results obtained include the above-mentioned results of Ya. I. Zhitomirskii and S. D. Eidelman. In studying uniqueness we shall proceed from the simple fact of uniqueness of the solution of the differential equation $du/dt = A(t)u$ in a Hilbert space with a nonnegative operator $A(t)$. The proof of existence is based on integral inequalities of the type of coercivity inequalities. The main difficulty encountered here is connected with proving a certain analogue of self-adjointness for an elliptic operator (see the lemma).

1°. In equation (1) we make the substitution $u = qv$ ($v = v(x,t)$ is the new unknown function, $q = q(x,t)$ is a fixed smooth positive function) and then multiply it by q^{-1}, after which it passes into a new equation with respect to v, which, for convenience, we write in the following form, separating the symmetric and skew-symmetric parts:

$$\frac{\partial v}{\partial t} = \sum_{i,j=1}^{N} D_i a_{ij}^{x,t} D_j v + 2\sum_{k=1}^{N} \tilde{b}_k(x,t) D_k v + \left(\sum_{k=1}^{N} D_k \tilde{b}_k(x,t)\right) v - \tilde{c}(x,t)v. \tag{1'}$$

By $A_0(t)$ and $A(t)$ we shall denote respectively the minimal and maximal operators (^{5}) constructed from the differential expression

$\mathfrak{M}_t$, standing on the right-hand side of equation $(1')$. Consider the following differential equation in $L_2(E_N)$:

$$dv/dt = A(t)v, \quad v|_{t=0} = v_0 \ (= q^{-1}(x,0)u_0(x)). \tag{3}$$

If, for equation (1), one can choose the function q so that, for equation (3), the fact of uniqueness (or existence) of a solution holds, then thereby we shall prove uniqueness (or existence) of a solution of equation (1) in the class of functions satisfying conditions of the type ($t \in [0,T]$)

$$\int q^{-2}(x,t)|u(x,t)|^2\,dx < \infty, \quad \int q^{-2}(x,t)|u'_t(x,t)|^2\,dx < \infty.$$

Let $g = g(x)$ be a continuous positive function, $k = \text{const} > 0$. We introduce into consideration the space $\mathcal{E}_g^k$, consisting of measurable functions $u(x)$, $x \in E_N$, such that

$$\int e^{-kg(x)}|u(x)|^2\,dx < \infty.$$

Everywhere in what follows, without specifying this each time separately, we shall assume that the coefficients of equation (1) are continuous in (x,t), and that $a_{ij}(x,t)$ and $b_k(x,t)$ are three times continuously differentiable with respect to x. Further, by $M(r)$ and $m(r)$, $0 \le r < \infty$, we shall denote such continuous positive functions that, for $t \in [0,T]$,

$$M(|x|)\sum \xi_k^2 \geq \sum a_{ij}(x,t)\xi_i\xi_j \geq m(|x|)\sum \xi_k^2.$$

2°. **Uniqueness.** We shall say that, for the Cauchy problem (1)–(2), uniqueness holds in the class $\mathcal{E}_g$, if there is no nonzero smooth solution $u(x,t)$ of equation (1) such that $u(x,0)=0$, $u(x,t)\in\mathcal{E}_g^k$, $u_t'(x,t)\in\mathcal{E}_g^k$ for some $k>0$ $(0\leq t\leq T)$.

Theorem 1. Suppose

$$\int_0^\infty M^{-1/2}(r)\,dr=\infty$$

and

$$\text{a)}\ |b_k(x,t)|\leq c_1\sqrt{M(r)}\,Q(r),$$

$$|c(x,t)|\leq c_2Q(r),\quad |D_ia_{ij}(x,t)|\leq c_3\sqrt{M(r)}\,Q(r),\quad |D_{iD}ja_{ij}(x,t)|\leq c_4Q(r),$$

$$|D_kb_k(x,t)|\leq c_5Q(r),\quad \text{where}\quad Q(r)=\left(\int_0^r M^{-1/2}(s)\,ds+1\right)^2.$$

Then, for the Cauchy problem (1)–(2), uniqueness holds in the class $\mathcal{E}_g$ with $g(x)=Q(r)$ $(r=|x|)$.

Theorem 2. Let $M(r)$ be arbitrary and suppose:

$$\text{a)}\ -c(x,t)-\frac{1}{2}\sum_{k=1}^N D_kb_k(x,t)+c_0\geq h(r)\geq 1$$

with some constant c_0;

$$\text{b)}\ |b_k(x,t)|\leq c_1\sqrt{h(r)M(r)};\quad |D_ia_{ij}(x,t)|\leq c_2\sqrt{h(r)M(r)};\quad |D_{iD}ja_{ij}(x,t)|\leq c_3h(r)\quad (r=|x|);$$

$$\text{c)}\quad \lim_{R\to\infty} h^{1/2}(R)\int_R^\infty M^{-1/2}(r)\,dr=\infty.$$

Then, for the Cauchy problem (1)–(2), uniqueness holds in the class $\mathcal{E}_g$ with

$$g(x)=\int_0^r h^{1/2}(s)M^{-1/2}(s)\,ds.$$

Example 1. Let $M(r) = O(r^\alpha)$, $\alpha < 2$. Then $Q(r) \geq C(1+r)^{2-\alpha}$, $C > 0$, and, if $c(x,t) = O(r^{2-\alpha})$, $b_k = O(r)$, $D_i a_{ij} = O(r)$, $D_k b_k = O(r^{2-\alpha})$, $D_{iD} j a_{ij} = O(r^{2-\alpha})$ as $r \to \infty$, then the solution of the Cauchy problem (1)–(2) is unique in the class of functions satisfying the condition

$$|u(x,t)| \leq c e^{kr^{2-\alpha}}.$$

In particular, for $\alpha = 0$ we obtain the result of Ya. I. Zhitomirskii and S. D. Eidelman (and even a more general one, since we do not require that $\inf_r m(r) > 0$).

Example 2. Let $M(r) = O(r^\alpha)$, $\alpha > 2$; $-c(x,t) \geq c_0(1+r)^\gamma + c_1 c_0 > 0$, $\gamma > \alpha - 2$; $b_k = O(r^{(\alpha+\gamma)/2})$, $D_k b_k = O(r^\gamma)$, $D_i a_{ij} = O(r^{(\alpha+\gamma)/2})$, $D_{iD} j a_{ij} = O(r^\gamma)$. The solution of the Cauchy problem (1)–(2) is unique in the class of functions satisfying the condition

$$|u(x,t)| \leq c \exp\left(kr^{(\alpha+\gamma+2)/2}\right), \quad k > 0.$$

Let us briefly outline the idea of the proof of Theorems 1 and 2. The conditions of these theorems make it possible to choose the function q so that in equation (1′) one has $\tilde{c} \geq 1$, whence the nonpositivity of the operator $A_0(t)$ follows: $\operatorname{Re}(A_0(t)v, v) \leq 0$ (here $q(x,t) = e^{s(x)\varphi(t)+\psi(t)}$, where $\varphi(t) > 0$, $\psi(t) > 0$). Next, the equality of the operators $A_0(t)$ and $A(t)$ is proved (see the lemma); establishing this fact is the main point of the entire proof. Thus, equation (1) is reduced to the operator equation (3) with a nonpositive operator $A(t)$, for which uniqueness of the solution is evident.

Lemma. *Suppose that for the elliptic differential expression with real coefficients*

$$\mathfrak{A}_1[u] = \sum_{i,j=1}^{N} D_i a_{ij}(x) D_j u + 2\sum_{k=1}^{N} b_k(x) D_k u + \left(\sum_{k=1}^{N} D_k b_k(x)\right) u - c(x)u$$

the following conditions are satisfied:

a) *the coefficients $a_{ij}(x)$, $b_k(x)$ are three times continuously differentiable, and $c(x)$ is continuous;*

b) $c(x) \geq 1$;

c)

$$\lim_{R\to\infty} C^{1/2}(R) \int_R^\infty \left(M_1(r) + \frac{B^2(r)}{C(r)}\right)^{-1/2} dr = \infty, \quad \text{where} \quad M_1(r) = \max_{i,j;\ |x|=r} |a_{ij}(x)|;$$

$$B(r) = \max_{k;\ |x|=r} |b_k(x)|; \quad C(r) = \min_{|x|=r} c(x).$$

Then the minimal operator A_0 *and the maximal operator* A*, constructed in* $L_2(E_N)$ *from the differential expression* $\mathfrak{A}_1$*, are equal to each other:* $A = A_0$*.*

This fact, for a self-adjoint expression with bounded coefficients, was established by Wienholtz (^{6}). A refinement of his proof makes it possible to dispense with the boundedness condition on the coefficients, which is replaced by the weaker condition c) of the lemma. We note that this condition, in the case of Theorem 1, is satisfied trivially, while in the case of Theorem 2 it follows from condition c) of the theorem.

3°. **Solvability.** By a generalized solution of the Cauchy problem (1)—(2) on $[0, \tau]$ we shall mean a function $u(x,t)$, locally summable in x and t ($x \in E_N$, $t \in [0,\tau]$), which, for every smooth function $\varphi(x,t)$ with compact support in x and with $\varphi(x,\tau) = 0$, satisfies the integral identity (obtained from equation (1) by integration by parts)

$$\int_0^\tau \int_{E_N} u(x,t)\,[\varphi_t' + \mathfrak{A}_t^+ \varphi]\,dx\,dt = -\int_{E_N} u_0(x)\varphi(x,0)\,dx. \tag{4}$$

Here $\mathfrak{A}_t^+$ is the differential operator formally adjoint to the differential operator $\mathfrak{A}_t$ appearing on the right-hand side of equation (1); the initial condition $u_0(x)$ is an arbitrary locally summable function. It is obvious that every smooth solution of the Cauchy problem (1)—(2) is at the same time generalized; conversely, if a smooth function $u(x,t)$ satisfies identity (4), then it also satisfies equation (1).

The reduction, carried out in the proof of Theorems 1 and 2, of equation (1) to the operator equation (3) with a nonpositive operator $A(t)$ makes it possible to establish the existence of a generalized solution of the Cauchy problem (1)—(2) (for $u_0 \in \mathcal{E}_g^2$), which, when the coefficients of the equation are sufficiently smooth, turns out to be ordinary by virtue of the hypoellipticity of the parabolic equation; moreover, it belongs (for each t) to the corresponding uniqueness class. However, in the general case we cannot prove uniqueness po-

of the solution obtained, since in Theorems 1 and 2 an essential role is played by the circumstance that not only the solution itself, but also its derivative with respect to t, belongs to the corresponding class.

To establish the correct solvability of the Cauchy problem, further restrictions must be imposed on the coefficients of the equation; these correspond to the following condition, which ensures the solvability of equation (3):

A. *The domain of definition of the operator* $A(t)$ *does not depend on* t*:* $D(A(t)) = D_0$*, and on* D_0 *the operator* $A(t)$ *has a weak bounded derivative* $A'(t)$ (^{3,7}).

We shall say that the Cauchy problem (1)—(2) is correctly solvable in the class $\mathcal{E}_g$ if, for every initial condition $u_0 \in \mathcal{E}_g^{k_0}$ (for some $k_0 > 0$, depending on u_0), there exists a unique generalized solution of the Cauchy problem $u(x,t)$, belonging for almost all t to $\mathcal{E}_g^k$ for some $k > 0$.

Theorem 3. *Suppose that the hypotheses of Theorem 1 are satisfied and, in addition:*

a) $m(r) \geq m_0 M(r)$, $m_0 > 0$;

b) the coefficients of equation (1) are continuously differentiable with respect to t;

c)

$$\left|\frac{\partial}{\partial t} a_{ij}(x,t)\right| \leq CM(r), \qquad \left|\frac{\partial}{\partial t} b_k(x,t)\right| \leq C\sqrt{M(r)}\,Q(r), \qquad \left|\frac{\partial}{\partial t} c(x,t)\right| \leq CQ(r).$$

Then the Cauchy problem (1)–(2) is correctly solvable on $[0,\tau]$ *in the class* $\mathcal{E}_g$ *with* $g = Q(r)$*, where* τ *depends on* k_0 *and on the coefficients of equation (1).*

Conditions a), b), c) of the theorem ensure the fulfillment of condition A; this follows from the following inequalities, valid for the operator $A(t)$ from equation (3) ($v(x)$ is a smooth finite function):

$$c_1\|v\|_1 \leq \|A(t)v\| \leq c_2\|v\|_1, \qquad c_1 > 0,$$

$$\|v\|_1^2 = \int M^2(r) \sum_{i,j=1}^{N} |D_{iD_{jv}}|^2\,dx + \int M(r)Q(r) \sum_{k=1}^{N} |D_{kv}|^2\,dx + \int Q(r) \times |v|^2\,dx.$$

Theorem 4. *Suppose that the hypotheses of Theorem 2 are satisfied and, in addition, the coefficients of the equation do not depend on* t*. Then the Cauchy problem (1)–(2) is correctly solvable on* $[0,T]$ *in the class* $\mathcal{E}_g$ *with*

$$g = \int_0^r h^{1/2}(s) M^{-1/2}(s)\,ds.$$

In this case condition A is fulfilled trivially, since in equation (3) the operator A does not depend on t. We note that one may also consider the case in which the coefficients of the equation depend on t; however, in that case new cumbersome conditions must be imposed on the coefficients.

The result of Ya. I. Zhitomirskii and S. D. Eidelman cited above is contained in Theorem 3 as a special case (for $M(r) = O(r)$); however, in essence these results coincide, since by a change of variables they can be reduced to one another.

We do not dwell here on the investigation of the smoothness of the generalized solution obtained, which can be carried out by known methods (see, for example, ([3])). In conclusion, we note that the results of Theorems 3–4 are also valid for an inhomogeneous equation under the corresponding conditions on the free term.

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Note: Figure translations are in progress. See original paper for figures.