THE METHOD OF MEAN DENSITY IN CALCULATING THE MOTION OF CHARGED PARTICLES ON ELECTRONIC COMPUTERS

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Abstract

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THE METHOD OF MEAN DENSITY IN CALCULATING THE MOTION OF CHARGED PARTICLES ON ELECTRONIC COMPUTERS

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Some authors ([1]) consider methods of direct integration of the dynamical equations in the many-body problem to be of little promise; the chief difficulties are the enormous number of equations and the need to specify initial values. However, the development of electronic computers makes it possible to hope that many problems will be solved precisely by this method. At present, the limited memory capacity of machines hinders the solution of the necessary number of equations, and therefore the development of direct methods will proceed along the path of reducing them. One such device is described in ([2]). Here another method is proposed, consisting in the following.

Let the equation of motion of charged particles be written in the form (17.5) ([3]), where the interaction is expressed through the Liénard–Wiechert potentials ([3]). Then, at the j-th observation point, the force acting on the charge e_j from the remaining particles, characterized by the density $\rho(x, y, z, t)$, is expressed in the form

$$\mathbf{a}_j = e_j \left(\vec{\varepsilon}_j + \frac{1}{c} [\dot{\mathbf{R}}_j \mathbf{h}_j] \right), \tag{1}$$

where the electric-field strength is

$$\vec{\varepsilon}_j = \int_{v-\Delta v_j} \rho \, dv \, \mathbf{b}_j,$$

the magnetic-field strength is

$$\mathbf{h}_j = \int_{v-\Delta v_j} \rho \, dv \, [\mathbf{R}, \mathbf{b}_j],$$

$$\mathbf{b}_j = \frac{1 - \dot{R}^2/c^2}{\left(R - \frac{\mathbf{R} \cdot \dot{\mathbf{R}}}{c}\right)^3} \left(\mathbf{R} - \frac{\mathbf{R}}{c} R \right) + \frac{\left[\mathbf{R} \left[\left(\mathbf{R} - \frac{\dot{\mathbf{R}}}{c} R \right) \ddot{\mathbf{R}} \right] \right]}{c^2 \left(R - \frac{\mathbf{R} \cdot \dot{\mathbf{R}}}{c} \right)^3},$$

the integral being taken over the entire volume of the charges except for the volume Δv_j of the charge itself; c is the speed of light; $\mathbf{R}$ is the radius vector; $\dot{\mathbf{R}} = d\mathbf{R}/dt$; $\ddot{\mathbf{R}} = d^2\mathbf{R}/dt^2$.

In computing the integral, the region in which the charges exist is divided by a grid with a uniform step in z ($\delta z_k = (z_{\max} - z_{\min})/K$), in x ($\delta x_i = (x_{\max} - x_{\min})/I$), and in y ($\delta y_p = (y_{\max} - y_{\min})/P$), with axes parallel to the coordinate axes and with the origin at the observation point. The density $\rho(x, y, z, t)$ is computed from the configuration of a finite number of trial particles ("observation points"), whose motion is found from the dynamical equations by solution on the machine, with forces determined by this density at preceding instants of time. For this purpose we introduce a trial cube $\Delta x \Delta y \Delta z = \Delta V$, containing several cubes $\delta x \cdot \delta y \cdot \delta z = \Delta v$ of the mesh used to partition the integral in (1), and define $\rho(x, y, z, t)$ as the ratio of the number of observation points in ΔV at time t to their number at $t = 0$.

The expression $\mathbf{b}_j$ is taken as the mean over the volume of the trial cube and is referred to its center, which moves during the integration according to the index-

by the indices k, i, p. In the nonuniform case, at $t = 0$ the density of the observation points is specified proportional to its value.

The most successful choice of the integration mesh, of the volume of the trial cube, and of the initial configuration of the observation points can be made only after concrete computations and depends on the size of the initial density charges and on the external field. The method does not make it possible to take into account processes associated with spatial periods in the density distribution smaller than the mesh period. It is obvious that as the numbers K, I, P and the initial density of observation points n_j are increased and ΔV is decreased, the solution of the problem approaches the true one; and in the limit this method and (2) will give one and the same result.

Let us consider, as an example, the motion of particles in a synchrocyclotron. A study performed for a single particle does not explain the capture of particles into the acceleration regime. We write the equation of motion of the j-th observation point in a cylindrical coordinate system in the form (4)

$$\ddot{r}_j = A_0(1-\beta_j^2)^{1/2}\left[A_{r_j}(1-\dot{r}_j^2) - r_j\dot{\theta}_j A_{\theta_j}\right] + \frac{\alpha_j^2}{r_j}, \tag{2}$$

$$\ddot{\theta}_j = \frac{1}{r_j}\left\{A_0(1-\beta_j^2)^{1/2}\left[A_{\theta_j}(1-\alpha_j^2) - \dot{r}_j\alpha_j A_{r_j}\right] - \frac{2\dot{r}_j\alpha_j}{r_j}\right\};$$

$$\ddot{z}_{jk} = A_0(1-\beta_j^2)^{1/2} A_{z_{jk}}(1-\dot{z}_{jk}^2), \tag{3}$$

where

$$A_0 = \frac{e_j}{m_{0j}c^2}; \qquad \alpha_j = r_j\dot{\theta}_j; \qquad \beta_j^2 = \dot{r}_j^2 + (r_j\dot{\theta}_j)^2; \qquad H_z = H_0(1 - hr^2/2)$$

for $r < 10$ cm;

$$H_z = ar + b$$

for $r > 10$ cm;

$$H_r = -\partial H_z/\partial r \cdot z; \qquad E_r = E_y \sin\theta; \qquad E_\theta = E_y \cos\theta;$$

$$E_y = \frac{\varepsilon_0 l^2}{l^2 + r^2 \sin^2\theta} \cos(1 \pm \Delta)(1 - \gamma t)t; \qquad A_r = E_r + \varepsilon_r + \alpha H_z;$$

$$A_\theta = E_\theta + \varepsilon_\theta - \dot{r} H_z; \qquad A_z = \varepsilon_z - \alpha H_r;$$

e_j and m_j are the charge and mass of the particle; ε_0 is the amplitude of the intensity of the accelerating field; H_r and H_z are the components of the magnetic field; H_0, h, a, b, l, Δ, and γ are constants of the synchrocyclotron. Neglecting $\ddot{R}, \dot{R}$ and referring all quantities to a single instant of time (2), we determine the terms that take into account the interaction $\varepsilon_r, \varepsilon_z$, and ε_θ as the projections of the expression

$$\vec{\varepsilon} = \int \frac{\rho\,\mathbf{R}\,dv}{R^3} \tag{4}$$

onto the corresponding axes. Since (2) does not depend on z, then

$$\rho(r, z, \theta, t) = \rho_z(z, t)\rho_{r\theta}(r, \theta, t); \tag{4'}$$

$$\vec{\varepsilon} = \int_{r_{\min}}^{r_{\max}} \int_{\theta_{\min}}^{\theta_{\max}} \rho_{r\theta} \int_{z_{\min}}^{z_{\max}} \frac{\rho_z(z,t)\,\mathbf{R}\,dz}{R^3}\, r\,dr\,d\theta. \tag{5}$$

The separation of the density (4′) and the writing of (4) in the form (5) make it possible to greatly reduce the number of equations (2), (3) used and to reduce our problem to determining the motion of geometric manifolds with coordinates $r_j, \theta_j, z_{\min} \le z_{kj} \le z_{\max}$, which are "rods" of variable length with axes along z, where j is the index enumerating the rods in r and θ, and K is the number of points of the initial division of the rod. $\rho_{r\theta}(r, \theta, t)$ at any point r, θ is determined as the ratio of the number of observation points $n(t, r, \theta)$ lying inside the trial square $r\,\Delta r\,\Delta\theta$ to its initial filling $n(0, r, \theta)$, and (5) is written as

$$\vec{\varepsilon}_j = B_0 \sum_{s=1}^{S} \frac{n(t, r, \theta)}{n(0, r, \theta)}\, r_s \delta r_s \delta\theta_s \int_{z_{\min}}^{z_{\max}} \rho_z \frac{\mathbf{R}_s\,dz}{R_s^3}, \tag{5'}$$

Figure 1

Figure 1: Figure 1

where B_0 is a constant characterizing the charge density at the initial instant of time; S is the number of squares into which the integral is divided. $\rho_z(z)$

at any point is determined as the ratio of the number of observation points $n(t,z)$ in the test segment Δz to its initial filling $n(0,z)$, and the integral over z will be:

$$D_0 \sum_{k=1}^{K} \frac{n(t,z)}{n(0,z)} \frac{\mathbf{R}_{sk}}{R_{sk}^3} \delta z_k, \tag{6}$$

where D_0 is a constant characterizing the charge density at the initial instant of time; K is the number of intervals in the partition of the integral. At the same time, since $\rho_z = C_0 \sum \Delta z_{\mathrm{H}} = C_0 \sum \frac{\Delta z}{z(z_{\mathrm{H}})}$, the integral over z can be represented by the "enlarged charge" method as

$$C_0 \int_{z_{s\,\min}}^{z_{s\,\max}} \frac{\mathbf{R}(t, r_s, \theta_s, z(z_{\mathrm{H}}))}{R^3(t, r_s, \theta_s, z(z_{\mathrm{H}}))}\, dz_{\mathrm{H}} \tag{6'}$$

(C_0 is a constant determined by the initial charge density).

Fig. 1. Example of the dependence $z(t)$. Current $5 \cdot 10^{-5}$ A, $|z_0| = 0, 1, 2, 3, 4, 5$ cm. a —without interaction, $|z_0| = z_{\max} = 5$ cm; b —with $|z_0| = z_{\max} = 5$ cm, current $2.5 \cdot 10^{-5}$ A.

The solution according to (2), (3), and (4) was carried out on the BESM of the Academy of Sciences of the USSR for parameters close to those of the synchrocyclotron of the Joint Institute for Nuclear Research.

The following were taken as initial conditions:

$$t = t_0, \qquad \dot{z}_{0k} = 0, \qquad z_{0k} = z_{0\,\min} + \frac{z_{0\,\max} - z_{0\,\min}}{K}\, k, \qquad k = 0, 1, \dots, K; \tag{7}$$

$$\dot{r}_{0ip} = 0, \qquad r_{0ip} = r_{0\,\min} + \frac{r_{0\,\max} - r_{0\,\min}}{I}\, i, \qquad i = 0, 1, \dots, I; \tag{8}$$

$$\dot{\theta}_{0ip} = 1, \qquad \theta_{0ip} = \theta_{0\,\min} + \frac{\theta_{0\,\max} - \theta_{0\,\min}}{P}\, p, \qquad p = 0, 1, \dots, P; \quad ip = j; \quad PI = N.$$

Fig. 2

Figure 2: Fig. 2

In view of the cumbersomeness of (2), (3), (7), (8) and the difficulty of using large I, P, and K, a preliminary analysis was carried out.

To estimate the influence of the interaction in z, (3), (5), and (7) were solved, where the integral over z was taken according to formulas (6) and (6′), while $r_j(t)$ and $\theta_j(t)$ in (3) were determined as the solution of (2) with initial values for $I = 2$, $i = 1$, $P = 2$, $p = 1$, and with constant $\rho_{r\theta}(t)$, $r_{\min}$, $r_{\max}$, $\theta_{\min}$, and $\theta_{\max}$.

The following results were obtained:

1. There exists a limiting charge with a current of the order of $5 \cdot 10^{-5}$ A, capable of being retained without losses.

2. The frequency of vertical (in z) oscillations is somewhat reduced compared with the case when (6) $\equiv 0$ (see Fig. 1).

3. For calculating $z(t)$ with an accuracy of 5%, it is sufficient to set $\Delta z = 2\delta z$, $n(0, z) = 2$, $K = 11$, i.e., the amounts of work by the average-density method and by the "enlarged charge" method (2) may be different.

4. In our case these amounts turned out to be approximately equal, since the machine time for computing the integral in the average-density method with a smaller partition interval was compensated by the increase in the time for extracting the principal value of the improper integral in the "enlarged charge" method.

The next stage consisted in solving (2), (5), and (8), with oscillations in z neglected and $\rho_z(t) = \text{const}$, $z_{\min} = \text{const}$, and $z_{\max} = \text{const}$. As a result we obtained the following.

1. Radial-phase oscillations (in $r\theta$) lead to "phase failure" (Fig. 2), i.e., the phenomenon described in [2] is repeated. Autophasing limits the magnitude of the accelerated current. In the example shown in Fig. 2, the phase failure occurs at $t = 3000$, at the moment of intensive displacement of the observation points occupying an extreme position at the initial instant of time. Subsequently these particles (Fig. 2, 1 and 2) drop out of the acceleration regime.

Fig. 2. Example of the dependence $r(t)$ for a beam of particles in a synchrocyclotron (the beam contours are shown by the dotted line). 1 and 2—$r(t)$ for observation points that have dropped out of the acceleration regime

2. The value of the current at which phase failure sets in is below the limiting value determined for interaction in z, and is equal to 10^{-5} A.

3. The largest values of the parameters in the representation of the integral were: $r\,\Delta r\,\Delta\theta = 4\,r\,\delta r\,\delta\theta$, $n(0, r, \theta) = 4$, $N = 48$.
4. The abrupt changes in the beam shape at the moment of "phase failure," obtained by the enlarged-charge method, are smoothed out when the average-density method is used.
5. Because of the limited capabilities of the machine, it was not possible to achieve complete agreement between the solution by the average-density method and by the enlarged-charge method.

Because of the limited RAM of the machine, it was possible to solve the complete system (2), (3), (4), (6), (7), (8) only for $P = 4$, $I = 3$, and $K = 4$. It was not possible to check convergence of the solution under additional subdivision. It turned out that particle losses occur in two ways: 1) oscillations in z contribute to the phenomenon of "phase failure" ; 2) in turn, radial-phase oscillations lead to additional broadening of the beam in the z direction; 3) both of these phenomena are observed at a current of $5 \cdot 10^{-6}$ A.

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Note: Figure translations are in progress. See original paper for figures.