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Abstract

Full Text

Mathematics

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ON THE REDUCIBILITY OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS IN A NEIGHBORHOOD OF A QUASI-PERIODIC MOTION

(Presented by Academician A. N. Kolmogorov, 11 XI 1961)

§ 1. The qualitative study of the behavior of solutions of an analytic system of ordinary differential equations

$$\dot{x} = X(x) \quad (x = (x_1, \dots, x_n)) \quad (1)$$

usually begins with finding particular, especially simple, solutions: equilibrium positions and periodic trajectories. Then one studies the arrangement of integral curves in a neighborhood of these solutions, which sometimes makes it possible to draw important conclusions about the behavior of solutions as a whole.

It turns out that in the general case the arrangement of integral curves in a neighborhood is the same as for a certain linear system. Namely, let A be the matrix of the linear part of the system in a neighborhood of the equilibrium position O , and let μ be the vector of eigenvalues of A . Suppose the following is satisfied.

Condition 1. There exist $\omega, K > 0$ such that

$$|\mu_i - (\mu, k)| > K|k|^{-\omega} \quad \left(|k| = \sum_{i=1}^n k_i > 1 \right) \quad (2)$$

for every vector k with integer nonnegative components k_i . Then system (1) is reducible to a linear one: there exists an analytic change of variables $y = y(x)$ in a neighborhood of O , transforming system (1) into

$$\dot{y} = Ay. \quad (3)$$

The possibility of such a reduction under Condition 1 was recently established by C. L. Siegel ⁽¹⁾. This condition is satisfied for almost every (in the sense of Lebesgue measure) vector μ . Under more restrictive conditions, satisfied in a

certain region of the μ -space, reducibility was proved already in the dissertation of H. Poincaré ⁽²⁾.

In a neighborhood of a periodic solution the situation is analogous, and here the general case is reducibility. In particular, linear systems with periodic coefficients are reducible, as has been known since the time of Floquet and Lyapunov (see, for example, ⁽³⁾).

Next in complexity after a periodic solution is a quasi-periodic one. Suppose that system (1) has an invariant manifold T , analytically homeomorphic to an s -dimensional torus. Then on T one can introduce angular coordinates $\theta = (\theta_1, \dots, \theta_s)$ so that the variables x will be analytic functions on T of θ , with period 2π in each θ_α . Suppose further that on T system (1) is equivalent to the system

$$\dot{\theta} = \lambda \quad (\lambda = (\lambda_1, \dots, \lambda_s) \text{ are real constants}). \quad (4)$$

One says that the torus T is **filled with quasi-periodic trajectories** if $(\lambda, m) \neq 0$ for no nonzero integral vector $m = (m_1, \dots, m_s)$.

The present paper is devoted to the study of the question of reducibility of a system to a linear one in a neighborhood of an invariant torus T .

The reducibility of a linear system with conditionally periodic coefficients was established recently, under certain conditions, by A. E. Gelman ⁽⁴⁾. We shall assume that the linear part of the equations in variations in a neighborhood of T is reducible, and by a suitable change of variables we shall reduce the nonlinear system to a linear one. Thus, in the general case, a nonlinear system is reducible or irreducible simultaneously with the linear one. Therefore of interest is the still unsolved question whether a linear system with conditionally periodic coefficients is reducible in the general case.

The required change of variables will be sought by the Newton method, usual after the work of A. N. Kolmogorov ⁽⁵⁾. The functions sought are expanded in Taylor-Fourier series. A characteristic feature of our case is that all terms of fixed degree in the Taylor expansion are determined in a finite number of approximations.

§ 2. It can be shown that the torus T has a neighborhood in the space x , analytically homeomorphic to the direct product \widehat{T} with coordinates θ and the r -dimensional cube $K^r = \{y : |y_j| \leq 1\}$ ($r = n - s$, $y = (y_1, \dots, y_s)$, $1 \leq j \leq s$). In this neighborhood the analytic system (1) takes the form

$$\dot{y} = A(\theta)y + f(y; \theta); \quad \dot{\theta} = \lambda + \varphi(y; \theta), \quad (5)$$

where the matrix $A(\theta)$ and the vector functions $f(y; \theta)$ and $\varphi(y; \theta)$ are analytic in the domain $|\operatorname{Im} \theta| < \rho$, $|y| < R$ and are periodic in θ_α with period 2π (here $\rho > 0$, $R > 0$ are certain constants). The expansion of $f(y; \theta)$ in powers of y

begins with terms of second degree, while that of $\varphi(y; \theta)$ begins with terms of first degree.

We shall assume that the linear system

$$\dot{y} = A(\theta)y \quad (6)$$

is reducible, i.e., that there exist a matrix $B(\theta)$, analytic for $|\operatorname{Im} \theta| < \rho_1$ on the torus T , and a constant matrix A_0 , satisfying the equation

$$\dot{B}B^{-1} + BAB^{-1} = A_0.$$

Then the linear change $z = B(\theta)y$ will reduce (6) to the form $\dot{z} = A_0z$.

In the variables θ, z the system (1) has the form (5) with the constant matrix $A(\theta) = A_0$. Therefore in what follows we consider only the system

$$\dot{y} = Ay + f(y; \theta); \quad \dot{\theta} = \lambda + \varphi(y; \theta) \quad (7)$$

with constant matrix A .

Theorem. Let $\mu = (\mu_1, \dots, \mu_m)$, $m \leq r$, be a complete set of eigenvalues of the matrix A , corresponding to distinct eigenvectors. Suppose that for some $K > 0$, $\omega > 0$

$$|(k, \mu) - \varepsilon\mu_j + i(\lambda, l)| > K(|k| + |l|)^{-\omega} \quad (\varepsilon = 0, 1; j = 1, \dots, m; i^2 = -1) \quad (8)$$

for any integer vectors $k = (k_1, \dots, k_m)$, $l = (l_1, \dots, l_s)$;

$$|k| = \sum_{\alpha=1}^m k_\alpha > 1 + \varepsilon, \quad k_\alpha \geq 0; \quad |l| = \sum_{\beta=1}^s |l_\beta|.$$

Then system (7) is reducible to a linear one, i.e., there exists a transformation, analytic and analytically invertible in some neighborhood $|y| < R^*$, $|\operatorname{Im} \theta| < \rho^*$ of the torus T ,

$$y^* = y + g(y; \theta); \quad \theta^* = \theta + \psi(y; \theta) \quad (9)$$

such that in the indicated neighborhood system (7) is equivalent to the system

$$\dot{y}^* = Ay^*; \quad \dot{\theta}^* = \lambda. \quad (10)$$

Thus, for the reducibility of system (7) it is sufficient that condition (8) be satisfied. This condition imposes restrictions only on the eigenvalues μ of the

matrix A and on the frequencies λ of the conditionally periodic motion (4). For $\omega > n = r + s$, restriction (8) is violated (for all k at once) only on a set of Lebesgue measure zero in the space of all vectors μ, λ . Therefore irreducible systems (7) should be regarded as exceptional.

§ 3. We shall seek the transformation (9) by Newton's method. On substituting (9) into (7), (10) must result. For this the conditions

$$\begin{aligned} -Ag + f + g_{yAy} + g_{yf} + g_{\theta}\lambda + g_{\theta}\varphi &= 0, \\ \varphi + \psi_{yAy} + \psi_{yf} + \psi_{\theta}\lambda + \psi_{\theta}\varphi &= 0. \end{aligned} \quad (11)$$

must be satisfied. The first approximation g_1, ψ_1 to g, ψ will be found from the shortened system

$$\begin{aligned} -Ag_1 + f + g_{1y}Ay + g_{1\theta}\lambda &= 0, \\ \varphi + \psi_{1y}Ay + \psi_{1\theta}\lambda &= 0. \end{aligned} \quad (12)$$

As we shall see below, the discarded terms are of the order of the square of the remaining ones. In the coordinates

$$y_1 = y + g_1(y; \theta); \quad \theta_1 = \theta + \psi_1(y; \theta) \quad (13)$$

system (7) will take the form

$$\dot{y}_1 = Ay_1 + f_1(y_1; \theta_1); \quad \dot{\theta}_1 = \lambda + \varphi_1(y_1; \theta_1), \quad (14)$$

where, in view of (11) and (12),

$$f_1(y_1, \theta_1) = [g_{1y}f + g_{1\theta}\varphi]; \quad \varphi_1(y_1, \theta_1) = [\psi_{1y}f + \psi_{1\theta}\varphi], \quad (15)$$

and the arguments y, θ of the functions in square brackets, after the differentiations, must be expressed in terms of y_1, θ_1 according to (13).

System (14) has the form (7), and in the same way one can find the next approximations $y_2 = y_1 + g_2(y_1; \theta_1); \theta_2 = \theta_1 + \psi_2(y_1; \theta_1)$, and so on.

The proof of convergence is based on the following lemma*.

Lemma. Suppose that in system (7) the vector functions f, φ are given by expansions convergent in the domain $|y| < R, |\operatorname{Im} \theta| < \rho$:**

$$f(y; \theta) = \sum_{\substack{k \geq 0 \\ |k| \geq 2}} \sum_l f_{k,l} y^k e^{i(l, \theta)},$$

$$\varphi(y; \theta) = \sum_{\substack{k \geq 0 \\ |k| \geq 1}} \sum_l \varphi_{k,l} y^k e^{i(l, \theta)}, \quad (16)$$

where $|f| < MR$, $|\varphi| < M$. Define the change of variables (13) by means of equations (12). Suppose condition (8) is satisfied. Then for any δ satisfying the inequalities

$$(LCM)^{1/4\omega} < \delta < 10^{-3} \min(1, \rho)$$

* For technical convenience we restrict ourselves to the case where the matrix A is diagonal. Our theorem is also valid in the case where A has Jordan blocks.

** Here $k = (k_1, \dots, k_r)$; $l = (l_1, \dots, l_s)$; $|k| = \sum_{\alpha=1}^r |k_\alpha|$; $|l| = \sum_{\beta=1}^s |l_\beta|$;

$$y^k = y_1^{k_1} \dots y_r^{k_r}; \quad f_{k,l} = (f_{k,l}^1, \dots, f_{k,l}^r) \quad \text{and} \quad \varphi_{k,l} = (\varphi_{k,l}^1, \dots, \varphi_{k,l}^s)$$

are vector Fourier-Taylor coefficients; $|f| = \max_\alpha |f^\alpha|$; $|\varphi| = \max_\beta |\varphi^\beta|$; $|y| = \max_\alpha |y_\alpha|$; $|\operatorname{Im} \theta| = \max_\beta |\theta_\beta|$.

(L is a constant depending only on n , C on n and K), the following assertions hold:

1. The transformation (13) is analytic for $|y| < Re^{-3\delta}$, $|\operatorname{Im} \theta| < \rho - 3\delta$, and in this domain $|g_1| < CMR\delta^{-2\omega}$, $|\psi_1| < CM\delta^{-2\omega}$, and the Jacobian J of the transformation (13) satisfies the inequalities

$$e^{-\delta} < J < e^\delta.$$

2. In the new coordinates (13), system (7) has the form (14), and in the domain $|y_1| < Re^{-4\delta}$, $|\operatorname{Im} \theta_1| < \rho - 4\delta$ the functions $f_1(y_1; \theta_1)$, $\varphi_1(y_1; \theta_1)$ are defined by convergent series of the form (16) and satisfy the inequalities

$$|f_1| < CR e^{-4\delta} \delta^{-4\omega} M^2; \quad |\varphi_1| < C \delta^{-4\omega} M^2. \quad (17)$$

3. The inverse transformation to (13), $y = y(y_1, \theta_1)$, $\theta = \theta(y_1, \theta_1)$, is defined for $|y_1| < Re^{-4\delta}$, $|\operatorname{Im} \theta| < \rho - 4\delta$ and maps this domain into a domain containing all points (y, θ) :

$$|y| < Re^{-5\delta}, \quad |\operatorname{Im} \theta| < \rho - 5\delta.$$

§ 4. A detailed proof of the lemma lies beyond the scope of the present note. The solutions g_1, ψ_1 of equations (12) are sought in the form

$$g_1 = \sum_{\substack{k \geq 0 \\ |k| \geq 2}} \sum_l g_{k,l} y^k e^{i(l,\theta)}; \quad \psi_1 = \sum_{\substack{k \geq 0 \\ |k| \geq 1}} \sum_l \psi_{k,l} y^k e^{i(l,\theta)}. \quad (18)$$

Substitution of (18) and (16) into (12) gives

$$g_{k,l}^j = - \frac{f_{k,l}^j}{(k, \mu) - \mu_j + i(l, \lambda)}; \quad \psi_{k,l} = - \frac{\varphi_{k,l}}{(k, \mu) + i(l, \lambda)}. \quad (19)$$

Estimating the small denominators in (19) with the aid of (8), after elementary calculations we obtain the estimates stated in the lemma. From (19) it is seen that g_1, ψ_1 are of the order of f, φ, M ; therefore, according to (15), f_1 and φ_1 are of order M^2 (see (17)). This ensures the rapid convergence of successive approximations.

If

$$|f| < R\Delta^{4\omega}(LC)^{-1}; \quad |\varphi| < \Delta^{4\omega}(LC)^{-1} = M, \quad (20)$$

where $\Delta = \min(1, \rho) \cdot 10^{-2}$, then one may pass to the limit and find the transformation (9), reducing (7) to the form (10).

But since in (7) $|f| = O(R^2)$, $|\varphi| = O(R)$, the conditions (20) are always satisfied for sufficiently small R . Therefore, in a sufficiently small neighborhood of the torus T , system (7) is reducible.

Let us note in conclusion that Siegel's theorem¹ on reducibility in a neighborhood of an equilibrium position and of a periodic solution is a special case of our theorem.

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Note: Figure translations are in progress. See original paper for figures.

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