MATHEMATICS

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Abstract

Full Text

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AN EQUATION OF NEUTRAL TYPE WITH A SMALL DELAY

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Consider the differential equation with deviating argument of neutral type ($\Delta t = \text{const} > 0$)

$$\dot{x}(t) = f(t, x(t), x(t - \Delta t), \dot{x}(t - \Delta t)) \qquad (1)$$

with the initial condition

$$x = \varphi(t), \qquad 0 \le t \le \Delta t. \qquad (2)$$

We shall study the behavior, as $\Delta t \to 0$, of the solution of (1) satisfying condition (2). In the case where (1) does not contain $\dot{x}(t-\Delta t)$ on the right (then it becomes an equation with delayed argument), as $\Delta t \to 0$ the solution has a limit, and this limiting function satisfies the equation obtained if, in the original equation, one formally sets $\Delta t = 0$, together with the initial condition $x = \varphi(0)$ at $t = 0$ (1). Moreover, under sufficient smoothness of the right-hand side of the equation and of the initial function $\varphi(t)$, for such a solution there is a valid asymptotic expansion in powers of Δt (2).

In the present note the main results of an investigation of the asymptotic properties of the solution of equation (1), determined by condition (2), as $\Delta t \to 0$, will be presented. This investigation shows that, under certain assumptions concerning the right-hand side of (1) (these assumptions include, besides the natural smoothness requirements, a special condition—the stability condition; see (3) below), for the solution $x(t, \Delta t)$ of equation (1) there is likewise a limiting transition as $\Delta t \to 0$, and the limiting function is constructed in the same way as in the above-mentioned case with delayed argument. In addition, asymptotic formulas will be written for x and $\dot{x}$, with a remainder term of order Δt^2, uniform over the entire interval of variation of t under consideration. At the same time, in a neighborhood of the initial point $t = 0$, the presence of a boundary layer is detected. In general, in the asymptotic behavior of the solution of equation (1), in the structure of the asymptotic formulas, and in the nature of a number of estimates, there is a great similarity with phenomena occurring for equations containing a small parameter multiplying the highest derivative (see, for example, (^{3},^{4})).

Let a continuous function $u = \Phi(t, x, y)$, which is a solution of the equation

$$u = f(t, x, y, u),$$

be defined in some rectangle $R : 0 \le t \le T,\ |x - \varphi(0)| \le b,\ |y - \varphi(0)| \le b$. Suppose that in the region $G : (t, x, y) \in R,\ |u - \Phi(t, x, y)| \le b$, the function f is continuous together with the partial derivative with respect to u, and that the condition

$$\left|\frac{\partial f}{\partial u}\right| \le a < 1, \tag{3}$$

is satisfied; we shall call this the stability condition. Assume that $\varphi(t)$ is continuously differentiable in a neighborhood of $t = 0$. Then the following holds:

Theorem 1. *If the above requirements are satisfied, then for sufficiently small* $\Delta t \le \Delta t_0$ *there exists a unique solution* $x(t, \Delta t)$ *of equation (1), satisfying condition (2); it is continuous in* t *and uniformly bounded with respect to* Δt *on the interval of variation of* $t : 0 \le t \le h$*, where*

$$h = \min\left(T, \frac{b - \delta}{K}\right),$$

$\delta > 0$ *is arbitrarily small, fixed as* $\Delta t \to 0$*, and*

$$K = \max_G |f|.$$

Theorem 2. Under the same assumptions, there is a limiting transition

$$\lim_{\Delta t \to 0} x(t, \Delta t) = \bar{x}(t), \qquad 0 \le t \le h, \tag{4}$$

where $\bar{x}(t)$ is a solution of the equation obtained from (1) by formally setting $\Delta t = 0$, satisfying the initial condition $\bar{x}(0) = \varphi(0)$.

In order to write asymptotic formulas for x and $\bar{x}$, we carry out some auxiliary constructions analogous to those done in (^{3},^{4}). Write the original equation (1) in the form of a system ($\dot{x} = u,\ [x] = x(t - \Delta t),\ [u] = u(t - \Delta t)$)

$$u = f(t, x, [x], [u]),$$

$$\frac{dx}{dt} = u; \tag{5}$$

$$x = \varphi(t), \qquad u = \dot{\varphi}(t), \qquad 0 \le t \le \Delta t. \tag{6}$$

Introducing a new variable $\tau = t/\Delta t$, we rewrite (5) also in the form

$$u = f(\tau \Delta t, x, [x], [u]),$$

$$\frac{dx}{d\tau} = \Delta t u. \tag{7}$$

We shall seek a formal solution of system (7) in the form of an expansion in powers of Δt (by z we shall denote x and u together)

$$z = z_0 + \Delta t z_1 + \cdots \tag{8}$$

Substituting this expansion into system (7), we obtain equations for determining z_0 and z_1 (to construct a formula with a remainder term of order Δt^2, coefficients of subsequent orders are not needed)

$$u_0 = f(0, x_0, [x_0], [u_0]) \equiv f_0,$$

$$\frac{dx_0}{dt} = 0; \tag{9}$$

$$\Delta t u_1 = f_{t0} t + f_{x0}(\Delta t x_1) + f_{y0}[\Delta t x_1] + f_{u0}[\Delta t u_1],$$

$$\frac{d}{dt}(\Delta t x_1) = u_0. \tag{10}$$

We prescribe the initial conditions for determining u_0, x_0 in the form

$$u_0 = \dot{\varphi}(0), \qquad 0 \le t \le \Delta t,$$

$$x_0|_{t=0} = \varphi(0). \tag{11}$$

Hence it follows that

$$x_0 \equiv \varphi(0),$$

$$u_0 = f(0, \varphi(0), \varphi(0), [u_0]), \tag{12}$$

$$u_0 = \dot{\varphi}(0), \qquad 0 \leq t \leq \Delta t.$$

We prescribe the initial conditions for determining u_1, x_1 in the form

$$\Delta t u_1 = t\ddot{\varphi}(0), \qquad 0 \leq t \leq \Delta t,$$

$$\Delta t x_1\big|_{t=0} = 0. \tag{13}$$

Now construct a formal solution of system (5) also in the form of an expansion in powers of Δt

$$z = \bar{z}_0 + \Delta t \bar{z}_1 + \cdots \tag{14}$$

In this case

$$\bar{u}_0 = f(t, \bar{x}_0, \bar{x}_0, \bar{u}_0) \equiv \bar{f}_0, \tag{15}$$

$$\dot{\bar{x}}_0 = \bar{u}_0.$$

To determine the solution of this equation, we impose the initial condition in the form

$$\bar{x}_0\big|_{t=0} = \varphi(0). \tag{16}$$

There is no need to prescribe an initial condition for $\bar{u}_0$. The functions $\bar{u}_1, \bar{x}_1$ are determined by the equations

$$\begin{aligned} &\bar{u}_1 = \bar{f}_{x0}\bar{x}_1 + \bar{f}_{y0}(\bar{x}_1 - \dot{\bar{x}}_0) + \bar{f}_{u0}(\bar{u}_1 - \dot{\bar{u}}_0), \\ &\dot{\bar{x}}_1 = \bar{u}_1. \end{aligned} \tag{17}$$

Here it is necessary to prescribe an initial condition for $\bar{x}_1$. We set $\bar{x}_1\big|_{t=0}$ equal to a certain constant, which we choose in the following special way. Denote

$$\dot{\varphi}(0) = a, \quad f(0, \varphi(0), \varphi(0), u) = Fu, \quad f(0, \varphi(0), \varphi(0), Fu) = F^2 u, \text{ etc.}$$

Consider the sequence

$$q_k = a + Fa + \cdots + F^{k-1}a - kF^{ka}, \tag{18}$$

which converges by virtue of condition (3). Put

$$\bar{x}_1\big|_{t=0} = \lim_{k\to\infty} q_k. \tag{19}$$

Denote $\bar{z}_0(0) = \bar{z}_{00}$, $\dot{\bar{z}}_0(0) = \bar{z}_{10}$, $\bar{z}_1(0) = \bar{z}_{01}$, and form the expressions

$$Z_0 = z_0 + \bar{z}_0 - \bar{z}_{00}; \tag{20}$$

$$Z_1 = z_0 + \Delta t\, z_1 + \bar{z}_0 + \Delta t\, \bar{z}_1 - (\bar{z}_{00} + t\bar{z}_{10} + \Delta t\, \bar{z}_{01}). \tag{21}$$

Theorem 3. *If $f(t, x, y, u)$ satisfies the stability condition (3) and has continuous partial derivatives in G up to order $n+2$ inclusive, and if $\varphi(t)$ is continuously differentiable $n + 2$ times in a neighborhood of $t = 0$, then for the solution z of system (5) satisfying the initial conditions (6), the inequalities*

$$|z - Z_n| < c\Delta t^{n+1} \quad (n = 0, 1), \tag{22}$$

hold, where c is a certain constant independent of t and Δt, provided only that $\Delta t < \Delta t_0$, where Δt_0 is sufficiently small, and $0 \le t \le h$.

On the interval $t_0 \le t \le h$, where t_0 is arbitrarily small but fixed as $\Delta t \to 0$, the inequalities

$$|z - \bar{z}_0| < c\Delta t, \qquad |z - (\bar{z}_0 + \Delta t\, \bar{z}_1)| < c\Delta t^2 \tag{23}$$

are valid.

Remark. In view of the fact that $x_0 = \varphi(0) = \bar{x}_0(0) = \bar{x}_{00}$, we have $X_0 \equiv \bar{x}_0$, and, consequently, (23) for x and $n = 0$ is valid not only on $t_0 \le t \le h$, but also on the entire interval $0 \le t \le h$.

The theorem thus asserts that the expressions (20), (21) serve, for the solution under study, as asymptotic formulas with remainder terms of orders Δt and Δt^2, respectively, which are uniform on the whole interval of variation of t under consideration. At the same time, inequalities (23) show that everywhere except in an arbitrarily small neighborhood of the initial point $t = 0$, simpler expressions, $\bar{z}_0$ and $\bar{z}_0 + \Delta t\, \bar{z}_1$, can serve as asymptotic formulas with the same degree of accuracy.

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Note: Figure translations are in progress. See original paper for figures.

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