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Abstract

Full Text

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On Singular Cauchy Problems for the Chaplygin Equation

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Let us consider, in the half-plane $\sigma > 0$, the Chaplygin equation:

$$z_{\theta\theta} - z_{\sigma\sigma} - b(\sigma) z_\sigma = 0, \qquad b(\sigma) = (\ln \sqrt{K(\sigma)})_\sigma, \qquad \sqrt{K} = \sum_{n=0}^{\infty} k_n \sigma^{(2n+1)/3} \quad (1)$$

and call its integrals $z(\theta, \sigma)$ and $\bar{z}(\theta, \sigma)$ solutions of the first and second singular Cauchy problems if, respectively $\left(\eta = -(\frac{3}{2}\sigma)^{2/3}\right)$,

$$z(\theta, 0) = \tau(\theta), \qquad z_\eta(\theta, 0) = \bar{z}(\theta, 0) = 0, \qquad \bar{z}_\eta(\theta, 0) = \nu(\theta). \quad (2)$$

As shown in paper (1), for z and $\bar{z}$ there are integral representations ($x = \theta - \sigma,\ y = \theta + \sigma$):

$$z(\theta, \sigma) = \int_x^y G(\theta - \alpha, \sigma) \tau(\alpha)\, d\alpha, \qquad \bar{z}(\theta, \sigma) = \int_x^y \bar{G}(\theta - \alpha, \sigma) \nu(\alpha)\, d\alpha, \quad (3)$$

where $G(\theta, \sigma)$ and $\bar{G}(\theta, \sigma)$ satisfy equation (1) and have the form $G = G_0 g(\theta, \sigma)$, $\bar{G} = \bar{G}_0 \bar{g}(\theta, \sigma)$, where $g(\theta, \sigma)$ and $\bar{g}(\theta, \sigma)$ are certain functions, bounded and nonzero on the characteristics $\theta \pm \sigma = 0$, while G_0 and $\bar{G}_0$ are the values of the kernels G and $\bar{G}$ for $b(\sigma) = \frac{1}{3\sigma}$: $G_0 = \gamma_1 (2\sigma)^{2/3} r^{-5/3}$, $\bar{G}_0 = -\gamma_2 r^{-1/3}$, $r = \sqrt{\sigma^2 - \theta^2}$, $\gamma_1 = \Gamma(1/3)/\Gamma^2(1/6)$, $\gamma_2 = (3/4)^{2/3} \Gamma(5/3)/\Gamma^2(5/6)$.

Divide the half-plane $\sigma \geq 0$ into four infinite angular regions $D_1 (0 \leq \sigma \leq \theta)$, $D_2 (0 \leq \theta \leq \sigma)$, $D_3 (0 \leq -\theta \leq \sigma)$, $D_4 (0 \leq \sigma \leq -\theta)$, formed by the lines $\theta = 0$, $\sigma = 0$, $\theta \pm \sigma = 0$ $(\sigma \geq 0)$. As in note (2), we shall agree to call the particular solutions $V(\theta, \sigma)$, $\bar{V}(\theta, \sigma)$, $W(\theta, \sigma)$, $\bar{W}(\theta, \sigma)$ of the problems (2) with discontinuous initial data the Duhamel functions:

$$V(\theta, 0) = \bar{V}_\eta(\theta, 0) = \tfrac{1}{2}(1 - \operatorname{sign} \theta); \qquad W(\theta, 0) = \bar{W}_\eta(\theta, 0) = \tfrac{1}{2} \operatorname{sign} \theta, \quad (4)$$

assuming that for all $-\infty \le \theta \le \infty$, $V_\eta(\theta,0) = W_\eta(\theta,0) = \bar{V}(\theta,0) = \bar{W}(\theta,0) \equiv 0$. Along with this, put in (1) and (2) $z = S(\sigma)$, $\tau(\theta) = \nu(\theta) = \mu$ ($\mu =$ const, $-\infty \le \theta \le \infty$), and introduce into consideration two integrals of the equation $z_{\sigma\sigma} + b(\sigma) z_\sigma = 0$:

$$S_\mu(\sigma) = \mu \quad \text{and} \quad \bar{S}_\mu(\sigma) = -\mu \int_0^\sigma \varkappa(\sigma)\, d\sigma,$$

$$\varkappa(\sigma) = k_0 (2/3)^{1/3} / \sqrt{K(\sigma)}.$$

Then it is not difficult to verify that the functions $V, \bar{V}, W$, and $\bar{W}$ possess the following properties:

1) They are related to $S_\mu(\sigma)$ and $\bar{S}_\mu(\sigma)$ by the equalities

$$V(\theta,\sigma) + W(\theta,\sigma) = S_{1/2}(\sigma); \qquad \bar{V}(\theta,\sigma) + \bar{W}(\theta,\sigma) = \bar{S}_{1/2}(\sigma). \tag{5}$$

2) In the closed region $\overline{D}_1$, $V = \bar{V} \equiv 0$, $W = S_{1/2}$, $\bar{W} = \bar{S}_{1/2}$, and at each point $(\theta,\sigma) \in \overline{D}_4$, $V = S_1$, $\bar{V} = \bar{S}_1$, $W = S_{-1/2}$, $\bar{W} = \bar{S}_{-1/2}$.

3) $W(\theta,\sigma)$ and $\bar{W}(\theta,\sigma)$ are odd functions of the variable θ, $W(0,\sigma) = \bar{W}(0,\sigma) \equiv 0$ and, consequently, by virtue of the equalities (5), $V(0,\sigma) = S_{1/2}$, $\bar{V}(0,\sigma) = \bar{S}_{1/2}$.

4) Substituting (4) into (3) and differentiating with respect to θ, we obtain in D_2 and D_3:

$$W_\theta(\theta,\sigma) = -V_\theta(\theta,\sigma) = G(\theta,\sigma); \qquad \overline{W}_\theta(\theta,\sigma) = -\overline{V}_\theta(\theta,\sigma) = \overline{G}(\theta,\sigma). \tag{6}$$

Formulas (6) show that the derivatives G and $\overline{G}$ of the odd functions W and $\overline{W}$ will be even functions of the variable θ, and therefore, in order to find them in $D_2 + D_3$, it suffices to compute the values of G and $\overline{G}$ only in the domain D_2. In turn, from conclusions 2) and 3) it follows that, for the construction of the resolvents $V, \overline{V}, W, \overline{W}$ in the domain D_2, it is sufficient to find integrals of equation (1) which on the characteristic $\theta = \sigma$ are equal to $V = \overline{V} = 0$, $W = S_{1/2}$, $\overline{W} = \overline{S}_{1/2}$, while on the line $\theta = 0$ ($\sigma \ge 0$), conversely, they take the values $V = S_{1/2}$, $\overline{V} = \overline{S}_{1/2}$, $W = \overline{W} = 0$. These boundary data are continuous for the functions $\overline{V}, \overline{W}$, and, conversely, in the case of V and W they undergo a discontinuity at the origin. Having obtained the solutions $V, \overline{V}$ or $W, \overline{W}$ of such mixed boundary-value problems, one can then easily determine, by formulas (6), the required singular kernels G and $\overline{G}$.

Let us carry out, for example, such computations for the functions V and $\overline{V}$. To this end we introduce in (1) the variables σ and $t = r/\sigma$ ($r = \sqrt{\sigma^2 - \theta^2}$),

$$\frac{1-t^2}{\sigma^2}\frac{\partial^2 z}{\partial t^2}-\frac{2(1-t^2)}{\sigma t}\frac{\partial^2 z}{\partial t\,\partial\sigma}-\frac{\partial^2 z}{\partial\sigma^2}+\frac{1-2t^2}{\sigma^2 t}\frac{\partial z}{\partial t}-b(\sigma)\left[\frac{1-t^2}{\sigma t}\frac{\partial z}{\partial t}+\frac{\partial z}{\partial\sigma}\right]=0 \tag{7}$$

and then set

$$z(\theta,\sigma)=V(\theta,\sigma)=\sum_{n=0}^{\infty}V_n(\theta,\sigma)=\sum_{n=0}^{\infty}\sigma^{2n/3}f_n(t). \tag{8}$$

Then, taking into account the expansion $b(\sigma)=\sum_{n=0}^{\infty}b_n\sigma^{2n/3-1}$ ($b_0=1/3,\ b_1=2k_1/3k_0,\ldots$), we arrive at the recurrent system of ordinary differential equations

$$L_n[f_n]\equiv t(1-t^2)f_n''+\frac{1}{3}[(4n-5)t^2+2-4n]f_n'-\frac{4}{9}n(n-1)tf_n$$

$$=\sum_{m=1}^{n}b_m\left[(1-t^2)f_{n-m}'+\frac{2}{3}(n-m)tf_{n-m}\right],\qquad f_{-1}(t)\equiv 0,\quad n=0,1,2,\ldots. \tag{9}$$

In addition, in order to ensure satisfaction of the boundary conditions $V(\sigma,\sigma)=0,\ V(0,\sigma)=1/2$, we set

$$f_0(0)=0,\quad f_0(1)=1/2;\qquad f_n(0)=f_n(1)=0\quad\text{for } n=1,2,\ldots. \tag{10}$$

Then first of all we find that

$$f_0(t)=\frac{1}{2}I_{t^2}\left(\frac{1}{6},\frac{1}{2}\right),$$

where

$$I_z(p,q)=\frac{z^pF(1-q,p,p+1;z)}{pB(p,q)}$$

is the modified incomplete beta function of Euler. Next we obtain

$$f_1(t)=\frac{3}{8}b_1\left[I_{t^2}\left(\frac{5}{6},\frac{1}{2}\right)-I_{t^2}\left(\frac{1}{6},\frac{1}{2}\right)\right].$$

Continuing this process, one can show that also for $n=2,3,\ldots$ the functions $f_2(t),f_3(t),\ldots$ are uniquely determined by conditions (9) and (10). Since $0\le I_z(p,q)\le 1$, if $0\le z\le 1$, then for $f_0(t)$ and $f_1(t)$ the estimates

$$0\le f_0(t)\le\frac{1}{2},\qquad |f_1(t)|\le\frac{3}{4}|b_1|\quad(0\le t\le 1)$$

hold ([3]).

Let us now turn to the construction of the Riemann function $\overline{V}(\theta,\sigma)$ of the second singular Cauchy problem. To this end we shall seek a solution of equation (7)

$$\bar{z} = \overline{V}(\theta,\sigma) = \sum_{n=1}^{\infty} \overline{V}_{n-1}(\theta,\sigma) = \sum_{n=1}^{\infty} \sigma^{2n/3} f_n(t) = \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^{n/3} (-\eta)^n f_n(t), \quad (11)$$

where, in order to satisfy the boundary conditions $\overline{V}(\sigma,\sigma) = 0$, $\overline{V}(0,\sigma) = \overline{S}_{1/2}(\sigma) = \sum_{n=1}^{\infty} A_n \sigma^{2n/3}$, where $A_1 = -\frac{1}{2}(3/2)^{2/3}$, $A_2 = -\frac{3}{4} b_1 A_1, \ldots$, we require that

$$f_n(0) = 0, \qquad f_n(1) = A_n \quad \text{for } n = 1, 2, \ldots. \quad (12)$$

Then in this case as well we arrive at system (9), in which, however, $f_0(t) \equiv 0$, while $f_1(t) = A_1 t^2 (5/6,\ 1/2)$ is uniquely determined from the homogeneous equation $L_1[f_1] = 0$ and equality (12). In addition, now for $n = 2$ the equation $L_2[f_2] = b_1[(1-t^2) f_1' + \frac{2}{3} t f_1]$ and condition (12) can be satisfied by setting $f_2(t) = -\frac{3}{4} b_1 f_1(t)$. In the same way, step by step, the subsequent solutions $f_3(t), f_4(t), \ldots$ are computed.

Restricting ourselves to the first two terms in the corresponding expansions for G and $\overline{G}$, we find, approximately, $G(\theta,\sigma) \cong P_2(\sigma) G_0 - \frac{1}{2}(3/2)^{1/3} b_1 \overline{G}_0$, $\overline{G}(\theta,\sigma) \cong P_2(\sigma)\overline{G}_0$, where $P_2(\sigma) = 1 - \frac{3}{4} b_1 \sigma^{2/3}$ is the partial sum of the series $\frac{2}{\eta}\overline{S}_{1/2}(\sigma) = \frac{2}{\eta}\sum_{n=1}^{\infty} A_n \sigma^{2n/3}$. With the aid of these estimates one can obtain the corresponding approximations for the Riemann function $v(\theta,\sigma,\theta_0,\sigma_0)$ of Chaplygin' s equation. Indeed, if instead of v we introduce the symmetric function $R(\theta,\sigma,\theta_0,\sigma_0) = \frac{1}{2}\chi(\sigma_0) v(\theta,\sigma,\theta_0,\sigma_0)$, then in a neighborhood $x_0 < x < y < y_0$ of the line $\sigma = 0$

$$R = \int_x^y \rho(\theta-\alpha,\sigma,\theta_0-\alpha,\sigma_0)\, d\alpha,$$

where

$$\rho(\theta,\sigma,\theta_0,\sigma_0) = G(\theta_0,\sigma_0)\overline{G}(\theta,\sigma) - G(\theta,\sigma)\overline{G}(\theta_0,\sigma_0).$$

Replacing here G and $\overline{G}$ by the approximations found, we obtain $R \cong P_2(\sigma)P_2(\sigma_0)R_0 + \frac{3}{4} b_1 \sigma^{2/3}[P_2(\sigma_0) - P_2(\sigma)]R_1$, where R_0 is the value of R for $K(\sigma) = k_0 \sigma^{1/3}$, and

$$R_1 = \gamma_2[(x-x_0)(y_0-x)]^{-1/6} F_1(5/6,\ 1/6,\ 1/6,\ 5/3;\ (y-x)/(x_0-x),\ (y-x)/(y_0-x)).$$

Analogous estimates can also be obtained in a neighborhood of the characteristics $x = x_0$, $y = y_0$, if one uses the integral representations found earlier ([4]) for the Riemann function in the domains $x_0 < x < y_0 < y$ and $x < x_0 < y < y_0$. In such representations the integrands contain initial values on the lines $\sigma = 0$, $\sigma_0 = 0$ of two modified Green–Hadamard functions $Q = \frac{1}{2}\chi(\sigma_0)H(\theta, \sigma, \theta_0, \sigma_0)$ and $\overline{Q} = \frac{1}{2}\chi(\sigma_0)\overline{H}(\theta, \sigma, \theta_0, \sigma_0)$, associated in D_1 with the boundary-value problems

$$z(\theta, 0) = \tau(\theta); \qquad z(\sigma, \sigma) = 0; \qquad \overline{z}_\eta(\theta, 0) = \nu(\theta); \qquad \overline{z}(\sigma, \sigma) = 0; \qquad \tau(0) = 0. \tag{13}$$

As follows from Gellerstedt's results, the solutions $z(\theta, \sigma)$ and $\overline{z}(\theta, \sigma)$ of problems (13) have the form

$$z(\theta, \sigma) = \int_0^x T(\theta - \alpha, \sigma)\tau(\alpha)\, d\alpha, \qquad \overline{z}(\theta, \sigma) = \int_0^x \overline{T}(\theta - \alpha, \sigma)\nu(\alpha)\, d\alpha,$$

where $T(\theta, \sigma) = T_0 q(\theta, \sigma)$, $\overline{T}(\theta, \sigma) = \overline{T}_0 \overline{q}(\theta, \sigma)$, with $q = O(1)$, $\overline{q} = O(1)$ as $\theta \to \sigma$, and $T_0 = \sqrt{3}\gamma_1(2\sigma)^{2/3} r_1^{-5/3}$, $\overline{T}_0 = \sqrt{3}\gamma_2 r_1^{-1/3}$, $r_1 = \sqrt{\theta^2 - \sigma^2}$. Similarly to the preceding case, it is also convenient here to apply the idea of preliminary integration of the kernels $T(\theta, \sigma)$, $\overline{T}(\theta, \sigma)$ with respect to the variable θ, and instead of these functions, which possess power singularities as $r_1 \to 0$, to consider the Doetsch resolvents U and $\overline{U}$, which are solutions of the problems (13) with discontinuous boundary data

$$U(\theta, 0) = 1, \qquad U(\sigma, \sigma) = 0, \qquad \overline{U}_\eta(\theta, 0) = 1, \qquad \overline{U}(\sigma, \sigma) = 0. \tag{14}$$

If, alongside them, we bring in the solutions $u(\theta, \sigma)$ and $\overline{u}(\theta, \sigma)$ of two conjugate discontinuous problems

$$u(\theta, 0) = 0, \qquad u(\sigma, \sigma) = S_1(\sigma); \qquad \overline{u}_\eta(\theta, 0) = 0, \qquad \overline{u}(\sigma, \sigma) = \overline{S}_1(\sigma), \tag{15}$$

then it is easy to verify that in the domain D_1

$$u + U = S_1, \quad \overline{u} + \overline{U} = \overline{S}_1, \quad U_\theta(\theta, \sigma) = -u_\theta(\theta, \sigma) = T(\theta, \sigma),$$

$$\overline{U}_\theta(\theta, \sigma) = -\overline{u}_\theta(\theta, \sigma) = \overline{T}(\theta, \sigma). \tag{16}$$

Thus, having constructed the functions $U, \overline{U}, u, \overline{u}$ from the boundary data (14) and (15), one can then, by formulas (16), also find the singular kernels $T, \overline{T}$. Such

The computations performed in (2) for U and $\overline{U}$ make it possible to determine, with prescribed accuracy, the initial data $Q_\eta|_{\sigma_0=0}$ and $\overline{Q}|_{\sigma_0=0}$ of the modified Green–Hadamard functions, and then from them, with the aid of formula (3), to construct the corresponding approximations for $Q(\theta,\sigma,\theta_0,\sigma_0)$ and $\overline{Q}(\theta,\sigma,\theta_0,\sigma_0)$. In turn, they make it possible to find the function R in a neighborhood of the incident and reflected characteristics from the transition line, and then also to solve more general mixed problems of the form (13) with nonzero data on the characteristic $\theta=\sigma$. The algorithms indicated above can also be applied in the study of other singular equations admitting a shift transformation in one of the coordinates. Of special interest, for example, is the equation $\sigma(z_{\theta\theta}-z_{\sigma\sigma})-b(\sigma)z_\sigma+c(\sigma)z=0$, if in a neighborhood of the line $\sigma=0$ $b(\sigma)$ and $c(\sigma)$ are represented by series in positive integral or fractional powers of σ, with $0<b(0)<1$, and $c(\sigma)=O(\sigma^\alpha)$ $(\alpha\geqslant 0)$. Equation (1) is also reduced to such a form if, by the substitution $z=\sigma^{1/6}K^{-1/4}u$, it is transformed into the form $u_{\theta\theta}-u_{\sigma\sigma}-\frac{1}{3\sigma}u_\sigma+c(\sigma)u=0$, where

$$c(\sigma)=\sum_{n=-1}^{\infty}c_n\sigma^{2n/3}.$$

In this connection, for the approximation of the solutions $u(\theta,\sigma)$, one may use as a standard not only the Euler–Poisson equation, but also the equation studied earlier[5]

$$u_{\theta\theta}-u_{\sigma\sigma}-\frac{a}{\sigma}u_\sigma-b^2u=0\qquad(0<a<1,\ b=\text{const}).\tag{17}$$

Here the equation $u_{\sigma\sigma}+\frac{a}{\sigma}u_\sigma+b^2u=0$, which determines $S_\mu(\sigma)$ and $\overline{S}_\mu(\sigma)$, is solved (under the conditions $S(0)=\overline{S}_\eta(0)=\mu$, $S_\eta(0)=\overline{S}(0)=0$, $\eta=-(\sigma/(1-a)^{1-a}))$ in Bessel functions $S_\mu(\sigma)=\mu\overline{J}_{\beta-1/2}(b\sigma)$, $\overline{S}_\mu(\sigma)=\mu\eta\overline{J}_{1/2-\beta}(b\sigma)$ $(a=2\beta)$, and therefore the corresponding expansions for V and $\overline{V}$ must have the form

$$V(\theta,\sigma)=\frac{1}{2}\sum_{n=0}^{\infty}\sigma^{2n}\varphi_n(t),\qquad \overline{V}(\theta,\sigma)=\frac{1}{2}\eta\sum_{n=0}^{\infty}\sigma^{2n}\overline{\varphi}_n(t).$$

They generate the systems

$$\mathcal{L}_{2n}[\varphi_n,a]=t(1-t^2)\varphi_n''+[1-a-4n+(4n+a-2)t^2]\varphi_n'-2n(2n+a-1)t\varphi_n=b^2t\varphi_{n-1},$$

$$\mathcal{L}_{2n}[\overline{\varphi}_n,2-a]=b^2t\overline{\varphi}_{n-1},\qquad(\varphi_{-1}=\overline{\varphi}_{-1}\equiv 0,\ n=0,1,2,\ldots),$$

to which, by virtue of the equalities $V(\sigma,\sigma)=\overline{V}(\sigma,\sigma)=0$, $V(0,\sigma)=S_{1/2}(\sigma)$, $\overline{V}(0,\sigma)=\overline{S}_{1/2}(\sigma)$, one should adjoin the conditions $\varphi_n(0)=\overline{\varphi}_n(0)=0$, $\varphi_n(1)=A_n$, $\overline{\varphi}_n(1)=\overline{A}_n$ $(n=0,1,2,\ldots)$, where $A_n=(-b^2/4)^n/n!(\beta+\frac{1}{2})_n$, and $\overline{A}_n$ is obtained from A_n by replacing β by $1-\beta$. Solving these boundary-value problems, we find the values $\varphi_n(t) = A_n I_{t^2}(n+\beta, \frac{1}{2})$, $\overline{\varphi}_n(t) = \overline{A}_n I_{t^2}(n+1-\beta, \frac{1}{2})$. They correspond to the double series

$$\overline{V} = \eta V(\theta, \sigma, 1-\beta),$$

$$V(\theta, \sigma, \beta) = \frac{\gamma_1}{\beta}\left(\frac{t}{2}\right)^a \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1/2)_m(\beta)_{m+n}}{m!\,n!\,(\beta)_n(\beta+1)_{m+n}}\, t^{2m}\left(-\frac{b^2r^2}{4}\right)^n.$$

However, these higher transcendental functions degenerate under differentiation with respect to θ and give the values

$$G = G_0\overline{J}_{\beta-1}(br), \qquad \overline{G} = \overline{G}_0\overline{J}_{-\beta}(br), \qquad G_0 = \gamma_1(2\sigma)^{1-a}r^{a-2},$$

$$\overline{G}_0 = -\gamma_2 r^{-a}, \qquad \gamma_1 = \frac{\Gamma(a)}{\Gamma^2(\beta)}, \qquad \gamma_2 = [2(1-a)]^{a-1}\frac{\Gamma(2-a)}{\Gamma^2(1-\beta)}.$$

Analogous constructions are also carried out for the example

$$z_{\theta\theta} - z_{\sigma\sigma} - \left(\frac{a}{\sigma} - 1\right) z_\sigma + \frac{c}{\sigma} z = 0 \qquad (0 < a < 1,\ c = \text{const}),$$

where

$$S_\mu(\sigma) = \mu_1 F_1(c, a, \sigma), \qquad \overline{S}_\mu(\sigma) = \mu\eta_1 F_1(c-a+1, 2-a, \sigma).$$

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Note: Figure translations are in progress. See original paper for figures.