MATHEMATICS

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

G. N. AGAEV

ON THE THEORY OF WEAK SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

(Presented by Academician S. L. Sobolev on 26 II 1962)

Let D be a bounded domain of the n-dimensional Euclidean space E_n. Consider the first boundary-value problem for a uniformly strongly elliptic system K of order $2m$ ($^{1-3}$)

$$Ku = \gamma, \tag{1}$$

$$u - g \in \overset{\circ}{W}{}_{2,r}^{(m)}(D), \tag{2}$$

where γ and g are given, u is the unknown, in general complex-valued r-vector functions (r-dimensional), defined in D; $\overset{\circ}{W}{}_{2,r}^{(m)}(D)$ is the Hilbert space obtained by completing the set $C_c^{\infty,r}(D)$ of infinitely differentiable finite r-vectors in D with respect to the metric of the S. L. Sobolev space $W_{2,r}^{(m)}(D)$ (^{4,5}).

Following Gårding (^{2}), by a weak solution of problem (1), (2) we shall mean a solution of the problem

$$(u \cdot \overline{K}\varphi) = (\gamma \cdot \varphi), \qquad u - g \in \overset{\circ}{W}{}_{2,r}^{(m)}(D) \tag{3}$$

for all $\varphi \in C_c^{\infty,r}(D)$, where $\overline{K}$ is the Lagrange adjoint of the operator K; $(\cdot)$ is the scalar product of the space $L_{2,r}(D)$:

$$(u \cdot v) = \sum_{i=1}^{r} \int_D u_i \overline{v}_i \, dx.$$

In what follows, unless otherwise stated, we shall assume everywhere that $u \in W_{2,r}^{(m)}(D)$ and $g \in W_{2,r}^{(m)}(D)$.

By Browder's theorem 1 (^{3}), if the coefficients of the system satisfy the conditions

$$a_{k_1 \ldots k_s;\, i,j}(x) \in C^{\sigma(s)}(\overline{D}) \cap C^s(D), \tag{4}$$

where $\sigma(s) = \max[0, s-m]$, $s = 0, 1, \ldots, 2m$, then the inequality

$$\mathrm{Re}(((-1)^{mK}\varphi\cdot\varphi)) \geq \rho\|\varphi\|_m^2 - k_0(\varphi\cdot\varphi) \tag{5}$$

holds for all $\varphi \in C_c^{\infty,r}(D)$, where $\|\ \|_m$ is the norm of the space $W_{2,r}^{(m)}(D)$, $\rho > 0$, $k_0 \geq 0$.

Following Browder (^{3}), introduce the space $H(D)$, consisting of elements of the set $\overset{\circ}{W}{}_{2,r}^{(m)}(D)$ with scalar product

$$(\varphi,\psi) = \mathrm{Re}(((-1)^{mK}\varphi\cdot\psi)) + k_0(\varphi\cdot\psi), \qquad \varphi,\psi \in C_c^{\infty,r}(D). \tag{6}$$

It is not difficult to show that the norms $\|\varphi\|_m$ and $\sqrt{(\varphi,\varphi)}$ are equivalent.

Theorem 1. If the system K satisfies conditions (4) and $\gamma \in L_{2,r}(D)$, then problem (3) is equivalent to the equation

$$[I + (-1)^m W]v - k_0 U v = \zeta \tag{7}$$

in $H(D)$, where W is a linear bounded skew-symmetric operator; U is a linear completely continuous self-adjoint operator; ζ is the element of $H(D)$ defined by

$$(\zeta,\varphi) = (\gamma\cdot\varphi) - (u_1\cdot\overline{K}\varphi), \qquad \varphi \in C_c^{\infty,r}(D), \tag{8}$$

where

$$u_1 = w_1 + g; \tag{9}$$

$$(w_1,\varphi) = (g\cdot\varphi), \qquad \varphi \in C_c^{\infty,r}(D), \qquad v = u + u_1. \tag{10}$$

It is not difficult to see that if $g = 0$, then from (9) and (10) we have $v = u$; if $g = 0$ and $\gamma = 0$, then from (8) it follows that $\zeta = 0$.

The proof is carried out by the generalized method of orthogonal projections, applied by Gårding to a single elliptic equation and by Browder to a strongly elliptic system.

On the basis of the Riesz theorem it is proved that if $u \in W_{2,r}^{(m)}(D)$, then

$$(u\cdot\varphi) = (Uu,\varphi), \qquad (u\cdot K^{(2)}\varphi) = -(Wu,\varphi) \tag{11}$$

for all $\varphi \in C_c^{\infty,r}(D)$, where U is a linear bounded self-adjoint operator; $K^{(2)} = \dfrac{K-\overline{K}}{2}$; W is a linear bounded skew-symmetric operator. From equality (11) the proof of Theorem 1 is obtained.

It is easily proved that the operator $I + (-1)^{mW}$ is invertible. Consequently, the Riesz-Schauder theory (^{6}) is applicable to equation (7).

Using the Riesz-Schauder theory, one can formulate the corresponding theorems on solvability and on the spectral property of problem (3). In particular, the following holds.

Theorem 2. Under conditions (4), if the homogeneous problem (3):

$$(u \cdot \overline{K}\varphi) = 0, \qquad \varphi \in C_c^{\infty,r}(D), \qquad u \in \overset{0}{W}{}_{2,r}^{(m)}(D)$$

has only the zero solution, then the nonhomogeneous problem (3) has a solution for arbitrary γ and g.

Let us note that for a domain D with sufficiently small diameter the homogeneous problem (3) has only the zero solution. For these diameters one can give an upper estimate. This estimate is determined by the coefficients of the system K.

On the basis of Browder' s theorem 2 (^{3}) on the regularity of the solution, the following assertion holds*: if instead of condition (4) one assumes the condition

$$a_{k_1 \ldots k_s; i,j}(x) \in C^{\sigma(s)+M,r}(\overline{D}) \cap C^{s,r}(D) \tag{12}$$

and assumes that $\gamma \in C^{M,r}(D) \cap L_{2,r}(D)$, where $M = (n+1)/2$ for odd n, $M = n/2$ for even n, then the weak solution of the boundary-value problem belongs to the class $C^{2m}(D)$, i.e. it is a solution of problem (1), (2) (a strong solution).

Every nontrivial solution of the problem

$$(u \cdot \overline{K}\varphi) = \lambda(u \cdot \varphi), \qquad \varphi \in C_c^{\infty,r}(D), \qquad u \in \overset{0}{W}{}_{2,r}^{(m)}(D) \tag{13}$$

will be called a weak eigenfunction, and the corresponding value of the parameter λ a weak eigenvalue of the operator K.

A number λ for which the problem

$$(u \cdot \overline{K - \lambda I}\varphi) = (\gamma \cdot \varphi), \qquad u - g \in \overset{0}{W}{}_{2,r}^{(m)}(D)$$

has a solution for all $\varphi \in C_c^{\infty,r}(D)$ and arbitrary γ and g, will be called a weak

—

* Let us note that many studies have recently been devoted to the question of regularity of solutions of the system K (^{10}).

regular. The set of numbers that are not weakly regular will be called the weak spectrum of the operator K. Under conditions (12), the weak spectrum of the operator K becomes the spectrum of the operator K.

Theorem 3. *The weak spectrum of the operator K consists of isolated weak eigenvalues with a point of accumulation, if it exists, at zero; the multiplicity of each weak eigenvalue is finite.*

From the second equality (11) it follows that if K is self-adjoint, i.e. $K = \overline{K}$, then $W = 0$; consequently, all weak eigenvalues of the operator K are real.

To find an approximate weak solution of the boundary-value problem (1), (2), i.e. of problem (3), we shall apply a method analogous to the Galerkin method.

Theorem 4. *Let problem (3) have a solution for arbitrary γ and g; let $v_i \in C_c^{\infty,r}(D)$, $i = 1, 2, \ldots$, be a complete orthonormal system in $H(D)$. Then the sequence*

$$\overset{n}{u} = \overset{n}{v} - u_1, \tag{14}$$

where

$$\overset{n}{v} = \sum_{k=1}^{n} \overset{n}{a}_k v_k, \tag{15}$$

tends strongly to the weak solution u of problem (3) in the space $\overset{\circ}{W}{}_{2,r}^{(m)}(D)$, where $\overset{n}{a}_k$, for sufficiently large n, are determined from the system

$$\sum_{k=1}^{n} \overset{n}{a}_k \, (K v_k \cdot v_j) = (\gamma \cdot v_j), \qquad j = 1, 2, \ldots, n. \tag{16}$$

The function u_1 is determined from (9).

The proof of this theorem is obtained on the basis of Theorem 1 and the theorem of S. G. Mikhlin (^{7}), applying it to equation (7).

On the basis of a theorem of N. I. Pol' skii (^{8}), the estimate

$$\left\| u - \overset{n}{u} \right\|_{W_{2,r}^{(m)}} \leq (1 + \varepsilon_n) \left\| v - \sum_{i=1}^{n} (v, v_i) v_i \right\|_{W_{2,r}^{(m)}(D)}, \tag{17}$$

holds, where $\varepsilon_n \to 0$ as $n \to \infty$.

We note that if the function u_1 is approximated by some sequence $\overset{n}{u}_1 \to u_1$, then the approximate value of the solution of the boundary-value problem (3) may be taken to be $\overset{n}{u} = \overset{n}{v} - \overset{n}{u}_1$. In particular, the sequence $\{\overset{n}{u}_1\}$ may be constructed from the functions $v_1, v_2, \ldots$, using the fact that w_1 in (9) is a function obtained as a result of applying Riesz' s theorem.

With problem (13) on weak eigenvalues we associate a spectral problem in an n-dimensional space:

$$(K\overset{n}{u} \cdot \varphi_j) = \overset{n}{\lambda}\,(\overset{n}{u} \cdot \varphi_j), \qquad j = 1, 2, \ldots, n. \tag{18}$$

The eigenvalues and eigenfunctions of (18) will be called, respectively, the approximate weak eigenvalues and weak eigenfunctions of problem (3).

On the basis of Theorem 2 of N. I. Pol' skii (^{8}), we have

Theorem 5. *All weak eigenvalues of the operator K, and only they, can be obtained as limits of all possible sequences of approximate weak eigenvalues; in other words, if λ_0 is a weak eigenvalue of the operator K, then from the aggregate*

of eigenvalues of the finite-dimensional systems (18) $^n\lambda_j,\ j = 1, 2, \ldots, n,\ n = 1, 2, \ldots$, one can select such a sequence $^n\lambda_{k_n},\ n = 1, 2, \ldots$, which converges to λ_0.

Theorem 6. Let λ_0 be a weak eigenvalue of the operator K. From any sequence $\{u^n_{k_n}\}$ of approximate weak eigenfunctions belonging to approximate weak eigenvalues $^n\lambda_{k_n} \to \lambda_0$, one can choose at least one strongly convergent subsequence, and every such subsequence converges to a weak eigenfunction of the operator K belonging to the weak eigenvalue λ_0.

Theorems 5 and 6 and Theorem 7 given below are proved as follows. From (13) we have $[I + (-1)^m W]u - k_0 Uu = \lambda Uu$ in the Hilbert space $H(D)$. The latter is equivalent to the equation

$$u = \lambda' RUu, \tag{19}$$

where $\lambda' = k_0 + \lambda,\ R = [I + (-1)^m W]^{-1}$. Consequently, the aforementioned theorem of N. I. Pol' skii (^{8}) is applicable to (19).

If A is a linear operator in an arbitrary Banach space, then the totality of elements x satisfying the equation

$$[I - \lambda_0 A]^k x = 0$$

for some k is called the invariant subspace corresponding to the eigenvalue λ_0.

On the basis of Theorem 4 of N. I. Pol' skii (^{8}) there holds

Theorem 7. All functions of the weak invariant subspace of the operator K corresponding to the weak eigenvalue λ_0, and only they, can be obtained as strong limits of all possible linear combinations of approximate invariant functions from approximate invariant subspaces corresponding to all approximate eigenvalues ${}^n\lambda_{k_n}$ that tend to λ_0.

We note that if $a_{k_1 \dots k_s;\,i,j}(x) \in C^{\sigma(s)+M,r}(\overline{D}) \cap C^{s,r}(D)$, then in Theorems 4–7 one may omit the word "weak," i.e., make the same assertions concerning the solution of the boundary-value problem (1), (2).

It is not difficult to see that all the theorems proved above are based on inequality (5). If on some dense set of functions inequality (5), or an analogue of it, is adopted as the fundamental property of an arbitrary operator K, then all our theorems remain valid for the corresponding problem for the operator K (see, for example, ([9])).

The author expresses deep gratitude to Academician of the Academy of Sciences of the Azerbaijan SSR Z. I. Khalilov for the formulation of the problem and for advice in carrying out this work.

Institute of Mathematics and Mechanics
Academy of Sciences of the Azerbaijan SSR

Received
21 II 1962

CITED LITERATURE

[1] M. I. Vishik, *Matem. sborn.*, **29**, no. 3, 615 (1951).
[2] L. Gårding, *Math. Scandinavica*, **1**, 237 (1953).
[3] F. Browder, *Contributions to the Theory of Partial Differential Equations*, Princeton, 1954.
[4] S. L. Sobolev, *Some Applications of Functional Analysis to Mathematical Physics*, L., 1950.
[5] V. I. Smirnov, *A Course of Higher Mathematics*, **5**, 1959.
[6] A. M. Kolmogorov, S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, M., 1954.
[7] S. G. Mikhlin, *Variational Methods in Mathematical Physics*, M., 1957.
[8] N. I. Pol' skii, *Ukr. matem. zhurn.*, **7**, no. 1 (1955).
[9] M. Schechter, *Sborn. per. Matematika*, **4**, 3, 1960.
[10] J. L. Lions, *Équations différentielles, opérationnelles et problèmes aux limites*, 1961.

Note: Figure translations are in progress. See original paper for figures.