MATHEMATICS

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Abstract

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ON ONE GENERAL FORMULATION OF THE PROBLEM OF BEST APPROXIMATION

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1. Let E_i $(i = 1, 2, 3)$ be a real space of type B; let H be a closed subspace of E_1; let A_i be a linear operator acting from H into E_{i+1} $(i = 1, 2)$.

In the space E_2 define a convex cone K, and for any two elements y_1, y_2 of E_2 agree to write $y_1 \geq y_2$ if $y_1 - y_2 \in K$.

Consider the following problem of best approximation of an element $x_0 \in E_1$.

Determine an element x^* at which the following is attained:

$$\inf \|x_0 - x\|, \tag{1}$$

where the lower bound is taken over elements x satisfying the constraints

$$x \in H; \tag{2}$$

$$A_1 x \geq b_1; \tag{3}$$

$$A_2 x = b_2. \tag{4}$$

Here $b_i \in E_{i+1}$ $(i = 1, 2)$. It is assumed that x_0 does not belong to the closure of the set of points x satisfying conditions (2)–(4).

The problem formulated contains, as a special case, the classical formulation of best approximation in a Banach space ([1],[2]), when only the constraint (2) is imposed on the element x. It is also a generalization of the problem of best approximation by polynomials whose coefficients are connected by a number of linear dependences (equalities and inequalities). Constraints of equality type correspond to restriction (4), and constraints of inequality type to condition (3).

2. In the present note an analogue of P. L. Chebyshev' s theorem is given for problem (1)–(4)—a necessary and sufficient condition for an element $x^* \in E_1$ to deliver the minimum in (1) under constraints (2)–(4). Introduce for consideration the spaces $\bar{E}_i$, conjugate to the spaces E_i $(i = 1, 2, 3)$. The positive cone $K \subset E_2$ induces a positive cone $\bar{K} \subset \bar{E}_2$: the convex cone $\bar{K}$ consists of elements $g \in \bar{E}_2$ such that $g(x) \geq 0$ for $x \in K$. If $g_1 - g_2 \in \bar{K}$, then we write $g_1 \geq g_2$. We shall say that the constraints (3), (4) of problem (1)–(4) have the property of regularity if: a) the operator A_2 maps H onto the entire space E_3; b) the cone K has nonempty interior and, for some x_1 satisfying constraints (2), (4), the element $A_1 x_1 - b_1$ is an interior point of the cone K.

Theorem 1. *Let conditions (3), (4) of problem (1)–(4) have the property of regularity, and let the element x^* be subject to constraints (2)–(4). In order that x^* deliver the minimum in (1) under constraints (2)–(4), it is necessary and sufficient that there exist functionals $g_i \in \bar{E}_i$ $(i = 1, 2, 3)$ satisfying the following conditions: a) $g_1(x_0 - x^*) = \|x_0 - x^*\|$, $\|g_1\| = 1$; b) $g_2 \geq 0$, $g_2 A_1 x^* = g_2 b_1$; c) $g_1(x) + g_2(A_1 x) + g_3(A_2 x) = 0$, $x \in H$.*

It should be noted that the regularity property which, by assumption, is satisfied by the constraints (3), (4), is used only in the proof of the necessity of the conditions of Theorem 1.

If the regularity property is violated, the sufficiency of the conditions of Theorem 1 is preserved; however, generally speaking, these conditions cease to be necessary. In all special formulations known to the author of problems of best approximation with additional conditions of the type of linear equalities or inequalities, the regularity conditions are satisfied. Therefore the analogue of P. L. Chebyshev' s theorem for each of these problems is a simple consequence of Theorem 1.

3. Let us give one important special case of problem (1)–(4). Let E be a real space of type B; let $x_0, x_1, \ldots, x_n$ be elements of E. We pose the problem of determining an element

$$x^* = \sum_{j=1}^{n} d_j^* x_j,$$

at which

$$\inf \left\| x_0 - \sum_{j=1}^{n} d_j x_j \right\| \tag{5}$$

is attained, where the lower bound is taken over all vectors $d = (d_1, d_2, \ldots, d_n)$ satisfying the constraints

$$\sum_{j=1}^{n} d_j a_j(t) \geq b(t), \qquad t \in D; \tag{6}$$

$$\sum_{j=1}^{n} d_j a_{ij} = b_i, \qquad i = 1, 2, \dots, r. \tag{7}$$

Here D is an arbitrary compact set; $b(t), a_j(t)$ $(j = 1, 2, \dots, n)$ are real-valued functions, continuous on D; the equations (7) are linearly independent.

It is assumed that the lower bound (5), subject to conditions (6), (7), is different from zero. The regularity property of the constraints (6), (7) of problem (5)–(7), which correspond to the constraints (3), (4) of problem (1)–(4), is formulated as follows:

There exists a vector $\bar{d} = (\bar{d}_1, \bar{d}_2, \dots, \bar{d}_n)$ satisfying the equalities (7) and turning all relations (6) into strict inequalities.

As a consequence of Theorem 1 we obtain an analogue of P. L. Chebyshev' s theorem for problem (5)–(7).

Theorem 2. *Let the constraints (6), (7) possess the regularity property, and let the vector* $d^* = (d_1^*, d_2^*, \dots, d_n^*)$ *be subject to these constraints.*

In order that the element $x^* = \sum_{j=1}^{n} d_j^* x_j$ *have the least deviation from* x_0 *under the conditions (6), (7), it is necessary and sufficient that there exist a functional* $g \in \bar{E}$ *and points* $t_i \in D$, $\sum_{j=1}^{n} d_j^* a_j(t_i) = b(t_i)$ $(i = 1, 2, \dots, s)$, *satisfying the following conditions: a)* $g(x_0 - x^*) = \|x_0 - x^*\|$, $\|g\| = 1$; *b)* $g(x_j) + \sum_{i=1}^{s} \beta_i a_j(t_i) + \sum_{i=1}^{r} \gamma_i a_{ij} = 0$ $(j = 1, 2, \dots, n)$; *c)* $\beta_i > 0$ *for* $i = 1, 2, \dots, s$; *d) the system of vectors* $\{a_1(t_i), a_2(t_i), \dots, a_n(t_i)\}$ $(i = 1, 2, \dots, s)$ *and* $(a_{i1}, a_{i2}, \dots, a_{in})$ $(i = 1, 2, \dots, r)$ *is linearly independent.*

For the case of the Chebyshev metric, problem (5)–(7) was considered in [3]. Theorem 1 of [3] is a special case of Theorem 2 of the present note.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.