# GEOPHYSICS

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## Abstract

## Full Text

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# A THREE-DIMENSIONAL PROBLEM ON THE FORECAST OF SMOOTHED VALUES OF METEOROLOGICAL ELEMENTS

In our paper ($^1$) we considered the problem of precomputing smoothed values of meteorological elements at the middle level of the atmosphere. Our investigation can be generalized to the general spatial case, when the meteorological elements depend essentially not only on horizontal coordinates and time, but also on the vertical coordinate.

We shall proceed from the model adopted in ($^2$). In this model the motion is regarded as quasi-solenoidal, and the problem reduces to determining a single function—the stream function—from the differential equation

$$\frac{\partial L}{\partial t} = \frac{1}{a_0^2 \sin\theta}(L, \psi) + 2\omega\Gamma\frac{\partial \psi}{\partial \lambda}. \tag{1}$$

Here $\psi(\theta, \lambda, \xi, t)$ is the stream function; $t$ is time; $\lambda$ is longitude of the point (increasing eastward); $\theta$ is the co-latitude (increasing southward); $\xi = \tilde{p}(z)/\tilde{p}(0)$, where $z$ is height above sea level, $\tilde{p}$ is standard pressure; $\omega$ is the angular velocity of the Earth' s rotation; $a_0$ is the radius of the Earth; $\Gamma$ is the mean, with respect to $\theta$ and $z$, value of the quantity

$$\frac{R^2 T_1}{4\omega^2 a_0^2 g}\left(\frac{\gamma_a - \gamma}{\cos^2\theta}\right);$$

$R$ is the gas constant; $T_1$ is the mean temperature of the atmosphere; $g$ is the acceleration due to gravity; $\gamma_a$ is the adiabatic lapse rate; $\gamma$ is the vertical temperature gradient. Moreover,

$$L = -\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \psi}{\partial \xi}\right) - \Gamma\Delta\psi, \qquad (L, \psi) = \frac{\partial L}{\partial \theta}\frac{\partial \psi}{\partial \lambda} - \frac{\partial L}{\partial \lambda}\frac{\partial \psi}{\partial \theta},$$

$$\Delta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}. \tag{2}$$

As boundary conditions we take (see ($^2$)):

$$\frac{\partial^2\psi}{\partial\xi\,\partial t} = -\frac{1}{a_0^2\sin\theta}\left(\psi, \frac{\partial\psi}{\partial\xi}\right) + W \quad \text{for } \xi = 1; \tag{3}$$

$$\xi\left(\frac{\partial^2\psi}{\partial\xi\,\partial t}\right) \text{ is bounded for } \xi = 0. \tag{4}$$

The quantity $W$ appearing in (3) is proportional to the vertical velocity at the upper boundary of the planetary boundary layer of the Earth and must be determined with the aid of some model of small-scale turbulence (for example, Akerblom’ s model)*.

We shall seek the smoothed values of the function $\psi$; denote them by $\overline{\psi}$. Deviations from the smoothed values will be denoted by $\psi'$:

$$\psi'(\theta,\lambda,\xi,t) = \psi(\theta,\lambda,\xi,t) - \overline{\psi}(\theta,\lambda,\xi,t).$$

* More precisely, condition (3) should be written not for $\xi = 1$, but at the boundary of the boundary layer; however, the difference in the final calculations would prove inessential.

Apply smoothing to equation (1); we obtain:

$$\frac{\partial\overline{L}}{\partial t} = -\frac{1}{a_0^2\sin\theta}\left[(\overline{L},\overline{\psi}) + \overline{(L',\psi')}\right] + 2\omega\Gamma\frac{\partial\overline{\psi}}{\partial\lambda}, \tag{5}$$

where

$$\overline{L} = -\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\overline{\psi}}{\partial\xi}\right) - \Gamma\Delta\overline{\psi} \quad \text{and} \quad L' = L - \overline{L}.$$

We may further write equation (1) in the form

$$\frac{\partial\overline{L}}{\partial t} + \frac{\partial L'}{\partial t} =$$

$$= -\frac{1}{a_0^2\sin\theta}\left[(\overline{L},\overline{\psi}) + (\overline{L},\psi') + (L',\overline{\psi}) + (L',\psi')\right] + 2\omega\Gamma\left(\frac{\partial\overline{\psi}}{\partial\lambda} + \frac{\partial\psi'}{\partial\lambda}\right). \tag{6}$$

Subtracting (5) from (6), we obtain the equation for the pulsations:

$$\frac{\partial L'}{\partial t} = -\frac{1}{a_0^2\sin\theta}\left[(\overline{L},\psi') + (L',\overline{\psi}) + (L',\psi') - \overline{(L',\psi')}\right] + 2\omega\Gamma\frac{\partial\psi'}{\partial\lambda}. \tag{7}$$

Write equation (7) for the points $M_1(\theta_1, \lambda_1, \xi_1)$ and $M_2(\theta_2, \lambda_2, \xi_2)$. Then multiply the equation written for $M_1$ by $L'(\theta_2, \lambda_2, \xi_2, t)$, and the equation written for $M_2$ by $L'(\theta_1, \lambda_1, \xi_1, t)$; adding the resulting equations and performing averaging, we arrive at the fundamental equation:

$$\frac{\partial \overline{L_2' L_1'}}{\partial t} = -\frac{1}{a_0^2 \sin\theta_1}\left[\overline{L_2'(\overline{L}_1, \psi_1')} + \overline{L_2'(L_1', \overline{\psi}_1)}\right] + 2\omega\Gamma\overline{L_2'\frac{\partial \psi_1'}{\partial \lambda_1}} +$$

$$+\frac{1}{a_0^2 \sin\theta_2}\left[\overline{L_1'(\overline{L}_2, \psi_2')} + \overline{L_1'(L_2', \overline{\psi}_2)}\right] + 2\omega\Gamma\overline{L_1'\frac{\partial \psi_2'}{\partial \lambda_2}}. \tag{8}$$

Here

$$\psi_i' = \psi'(\theta_i, \lambda_i, \xi_i, t), \qquad \overline{\psi}_i = \overline{\psi}(\theta_i, \lambda_i, \xi_i, t),$$

$$L_i' = -\frac{\partial}{\partial \xi_i}\left(\xi_i^2 \frac{\partial \psi_i'}{\partial \xi_i}\right) - \Gamma\Delta_i\psi_i', \qquad \overline{L}_i = -\frac{\partial}{\partial \xi_i}\left(\xi_i^2 \frac{\partial \overline{\psi}_i}{\partial \xi_i}\right) - \Gamma\Delta_i\overline{\psi}_i,$$

$$\Delta_i = \frac{1}{\sin\theta_i}\frac{\partial}{\partial\theta_i}\left(\sin\theta_i\frac{\partial}{\partial\theta_i}\right) + \frac{1}{\sin^2\theta_i}\frac{\partial^2}{\partial\lambda_i^2} \qquad (i = 1, 2).$$

In deriving (8), an essential limitation was the neglect of terms containing third powers of the pulsations. We now introduce the correlation moments $B_\psi^\psi$ from the equality

$$B_\psi^\psi = \overline{\psi'(\theta_1, \lambda_1, \xi_1, t)\psi'(\theta_2, \lambda_2, \xi_2, t)} = \overline{\psi_1'\psi_2'}$$

and transform (8) so that only the correlation moments $B_\psi^\psi$ and the mean values $\overline{\psi}$ enter this equation. First note that the expression $\overline{L_2'L_1'}$ can be written in the form

$$\overline{\left[\frac{\partial}{\partial \xi_1}\left(\xi_1^2\frac{\partial \psi_1'}{\partial \xi_1}\right) + \Gamma\Delta_1\psi_1'\right]\left[\frac{\partial}{\partial \xi_2}\left(\xi_2^2\frac{\partial \psi_2'}{\partial \xi_2}\right) + \Gamma\Delta_2\psi_2'\right]},$$

or, since $\theta_1, \lambda_1, \xi_1$ and $\theta_2, \lambda_2, \xi_2$ are independent variables, we may write

$$\overline{L_2'L_1'} = \left[\frac{\partial}{\partial \xi_1}\left(\xi_1^2\frac{\partial}{\partial \xi_1}\right) + \Gamma\Delta_1\right]\left[\frac{\partial}{\partial \xi_2}\left(\xi_2^2\frac{\partial}{\partial \xi_2}\right) + \Gamma\Delta_2\right]\overline{\psi_1'\psi_2'} = \mathcal{L}_1\mathcal{L}_2 B_\psi^\psi,$$

where the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ denote

$$\mathcal{L}_i = -\frac{\partial}{\partial \xi_i}\left(\xi_i^2 \frac{\partial}{\partial \xi_i}\right) - \Gamma\Delta_i \qquad (i = 1, 2).$$

The remaining terms of (8) are transformed analogously.

As a result of similar transformations we arrive at a single equation containing $B_\psi^\psi$ and $\bar{\psi}$:

$$\mathcal{L}_1\mathcal{L}_2 \frac{\partial B_\psi^\psi}{\partial t} = \frac{1}{a_0^2 \sin\theta_1}\left[\frac{\partial \bar{\psi}_1}{\partial \lambda_1}\frac{\partial}{\partial \theta_1}(\mathcal{L}_1\mathcal{L}_2 B_\psi^\psi) - \frac{\partial \bar{\psi}_1}{\partial \theta_1}\frac{\partial}{\partial \lambda_1}(\mathcal{L}_1\mathcal{L}_2 B_\psi^\psi) + \right.$$

$$\left. + \frac{\partial \bar{L}_1}{\partial \theta_1}\frac{\partial}{\partial \lambda_1}(\mathcal{L}_2 B_\psi^\psi) - \frac{\partial \bar{L}_1}{\partial \lambda_1}\frac{\partial}{\partial \theta_1}(\mathcal{L}_2 B_\psi^\psi)\right] + 2\omega\Gamma \frac{\partial}{\partial \lambda_1}(\mathcal{L}_2 B_\psi^\psi) +$$

$$+ \frac{1}{a_0^2 \sin\theta_2}\left[\frac{\partial \bar{\psi}_2}{\partial \lambda_2}\frac{\partial}{\partial \theta_2}(\mathcal{L}_1\mathcal{L}_2 B_\psi^\psi) - \frac{\partial \bar{\psi}_2}{\partial \theta_2}\frac{\partial}{\partial \lambda_2}(\mathcal{L}_1\mathcal{L}_2 B_\psi^\psi) + \right.$$

$$\left. + \frac{\partial \bar{L}_2}{\partial \theta_2}\frac{\partial}{\partial \lambda_2}(\mathcal{L}_1 B_\psi^\psi) - \frac{\partial \bar{L}_2}{\partial \lambda_2}\frac{\partial}{\partial \theta_2}(\mathcal{L}_1 B_\psi^\psi)\right] + 2\omega\Gamma \frac{\partial}{\partial \lambda_2}(\mathcal{L}_1 B_\psi^\psi). \qquad (9)$$

The second equation, containing $B_\psi^\psi$ and $\bar{\psi}$, is easily obtained directly from equation (5), in which it is only necessary to transform the expression for $(L', \psi')$.

We use the obvious equality

$$\frac{\partial L'}{\partial \theta}\frac{\partial \psi'}{\partial \lambda} = \frac{1}{2}\left(B_{\partial L/\partial\theta}^{\partial\psi/\partial\lambda} + B_{\partial\psi/\partial\lambda}^{\partial L/\partial\theta}\right)_{\substack{\theta_1=\theta_2=\theta,\ \lambda_1=\lambda_2=\lambda, \\ \xi_1=\xi_2=\xi}} =$$

$$= \frac{1}{2}\left[\frac{\partial^2}{\partial\theta_1\partial\lambda_2}(\mathcal{L}_1 B_\psi^\psi) + \frac{\partial^2}{\partial\lambda_1\partial\theta_2}(\mathcal{L}_2 B_\psi^\psi)\right]_{\substack{\theta_1=\theta_2=\theta,\ \lambda_1=\lambda_2=\lambda, \\ \xi_1=\xi_2=\xi}},$$

and the analogous equality for $\frac{\partial L'}{\partial \lambda}\frac{\partial \psi'}{\partial \theta}$. We obtain, instead of (5),

$$\frac{\partial \bar{L}}{\partial t} = \frac{1}{a_0^2 \sin\theta}(\bar{L}, \bar{\psi}) + 2\omega\Gamma \frac{\partial \bar{\psi}}{\partial \lambda} +$$

$$+ \frac{1}{2a_0^2 \sin\theta}\left[\frac{\partial^2}{\partial\theta_1\partial\lambda_2}(\mathcal{L}_1 B_\psi^\psi - \mathcal{L}_2 B_\psi^\psi) - \frac{\partial^2}{\partial\lambda_1\partial\theta_2}(\mathcal{L}_1 B_\psi^\psi - \mathcal{L}_2 B_\psi^\psi)\right]_{\substack{\theta_1=\theta_2=\theta,\ \lambda_1=\lambda_2=\lambda, \\ \xi_1=\xi_2=\xi}}. \qquad (10)$$

Let us now turn to the boundary condition (3). It may be written in the form

$$\frac{\partial^2 \bar\psi}{\partial\xi\,\partial t} + \frac{\partial^2 \psi'}{\partial\xi\,\partial t} =$$

$$= -\frac{1}{a_0^2 \sin\theta}\left[\left(\bar\psi, \frac{\partial\bar\psi}{\partial\xi}\right) + \left(\bar\psi, \frac{\partial\psi'}{\partial\xi}\right) + \left(\psi', \frac{\partial\bar\psi}{\partial\xi}\right) + \left(\psi', \frac{\partial\psi'}{\partial\xi}\right)\right] + \bar W + W'$$

$$\text{for } \xi = 1. \tag{11}$$

Averaging (11), we shall have

$$\frac{\partial^2 \bar\psi}{\partial\xi\,\partial t} = -\frac{1}{a_0^2 \sin\theta}\left[\left(\bar\psi, \frac{\partial\bar\psi}{\partial\xi}\right) + \overline{\left(\psi', \frac{\partial\psi'}{\partial\xi}\right)}\right] + \bar W \quad \text{for } \xi = 1. \tag{12}$$

Subtracting (12) from (11), we obtain the boundary condition for the pulsations

$$\frac{\partial^2 \psi'}{\partial\xi\,\partial t} = -\frac{1}{a_0^2 \sin\theta}\left[\left(\bar\psi, \frac{\partial\psi'}{\partial\xi}\right) + \left(\psi', \frac{\partial\bar\psi}{\partial\xi}\right) + \left(\psi', \frac{\partial\psi'}{\partial\xi}\right) - \overline{\left(\psi', \frac{\partial\psi'}{\partial\xi}\right)}\right] + W',$$

$$\text{for } \xi = 1. \tag{13}$$

Write condition (13) in the variables $\theta_1, \lambda_1, \xi_1$ and multiply by $(\partial\psi_2'/\partial\xi_2)_{\xi_2=1}$; then write (13) in the variables $\theta_2, \lambda_2, \xi_2$ and multiply by $(\partial\psi_1'/\partial\xi_1)_{\xi_1=1}$. Adding the obtained equalities, averaging the result (here we neglect third powers of the pulsations), and expressing all quantities through $B_\psi^\psi$ and $\bar\psi$

(analogously to how this was done in deriving equations (9) and (10)), we obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{\partial B_\psi^\psi}{dt}\right)_{\xi_1=\xi_2=1} = &-\frac{1}{a_0^2}\left\{\frac{1}{\sin\theta_1}\left[\frac{\partial\bar\psi_1}{\partial\theta_1}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{\partial B_\psi^\psi}{\partial\lambda_1} - \frac{\partial\bar\psi_1}{\partial\lambda_1}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{\partial B_\psi^\psi}{\partial\theta_1} + \frac{\partial^2\bar\psi_1}{\partial\lambda_1\partial\xi_1}\frac{\partial^2 B_\psi^\psi}{\partial\theta_1\partial\xi_2} - \frac{\partial^2\bar\psi_1}{\partial\theta_1\partial\xi_1}\frac{\partial^2}{\partial\lambda_1}\right.\right.\\ &+\frac{1}{\sin\theta_2}\left[\frac{\partial\bar\psi_2}{\partial\theta_2}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{\partial B_\psi^\psi}{\partial\lambda_2} - \frac{\partial\bar\psi_2}{\partial\lambda_2}\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{\partial B_\psi^\psi}{\partial\theta_2} + \frac{\partial^2\bar\psi_2}{\partial\xi_2\partial\lambda_2}\frac{\partial^2 B_\psi^\psi}{\partial\theta_2\partial\xi_1}\right.\\ &\left.\left.-\frac{\partial^2\bar\psi_2}{\partial\xi_2\partial\theta_2}\frac{\partial^2 B_\psi^\psi}{\partial\lambda_2\partial\xi_1}\right]\right\}_{\xi_1=\xi_2=1} + \left(\frac{\partial\psi_1'}{\partial\xi_1}W_2' + \frac{\partial\psi_2'}{\partial\xi_2}W_1'\right)_{\xi_1=\xi_2=1}.\end{aligned} \tag{14}$$

Finally, in condition (12), let us express $\left(\psi', \frac{\partial\psi'}{\partial\xi}\right)$ in terms of correlation moments (using transformations analogous to those by means of which equation (10) was obtained). We obtain the second boundary condition in the form

$$\left(\frac{\partial^2\bar{\psi}}{\partial\xi\,\partial t}\right)_{\xi=1} = -\frac{1}{a_0^2\sin\theta}\left\{\left(\bar{\psi}, \frac{\partial\bar{\psi}}{\partial\xi}\right)_{\xi=1} + \frac{1}{2}\left[\frac{\partial^2}{\partial\theta_1\partial\lambda_2}\left(\frac{\partial B_\psi^\psi}{\partial\xi_2} - \frac{\partial B_\psi^\psi}{\partial\xi_1}\right) - \frac{\partial}{\partial\lambda_1}\frac{\partial}{\partial\theta_2}\left(\frac{\partial B_\psi^\psi}{\partial\xi_2} - \frac{\partial B_\psi^\psi}{\partial\xi_1}\right)\right]\right\}_{\substack{\theta_1=\theta_2=\theta,\ \lambda_1=\lambda_2=\lambda,\\ \xi_1=\xi_2=1}} + \bar{W}. \tag{15}$$

The solution of the problem can be carried out in time steps, determining $\partial\bar{\psi}/\partial t$ from equation (10) under boundary condition (15), and finding $\partial B_\psi^\psi/\partial t$ from equation 9) under boundary condition (14). In determining $\partial\bar{\psi}/\partial t$, the problem reduces to solving an equation of the form $\mathcal{L}\,\frac{\partial\bar{\psi}}{\partial t} = F(\theta, \lambda, \xi)$ under the boundary condition $\left(\frac{\partial}{\partial\xi}\frac{\partial\bar{\psi}}{\partial t}\right)_{\xi=1} = f(\theta, \lambda)$, where the operator $\mathcal{L} = -\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial}{\partial\xi}\right) - \Gamma\Delta$, and $F$ and $f$ are known functions. This problem was solved in ($^2$).

To determine $\partial B_\psi^\psi/\partial t$, we have an equation of the form

$$\mathcal{L}_1\mathcal{L}_2\frac{\partial B_\psi^\psi}{\partial t} = \Phi(\theta_1, \lambda_1, \xi_1; \theta_2, \lambda_2, \xi_2)$$

under the boundary condition

$$\frac{\partial^2}{\partial\xi_1\partial\xi_2}\frac{\partial B_\psi^\psi}{\partial t} = \varphi(\theta_1, \lambda_1; \theta_2, \lambda_2) \quad \text{for} \quad \xi_1 = \xi_2 = 1,$$

where $\Phi$ and $\varphi$ are known functions (the problem reduces to a twofold inversion of the operator $\mathcal{L}$). In doing so, it must be remembered that

$$B_\psi^\psi(\theta_1, \lambda_1, \xi_1; \theta_2, \lambda_2, \xi_2; t) \equiv B_\psi^\psi(\theta_2, \lambda_2, \xi_2; \theta_1, \lambda_1, \xi_1; t).$$

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## REFERENCES CITED

[1] E. N. Blinova, DAN, 123, No. 3, 440 (1958). [2] E. N. Blinova, DAN, 110, No. 6, 975 (1956).

* If $W = k(\Delta\psi)_{\xi=1}$, then

$$\left(\frac{\partial \psi_1'}{\partial \xi_1} W_2' + \frac{\partial \psi_2'}{\partial \xi_2} W_1'\right)_{\xi_1=\xi_2=1} = k\left(\frac{\partial \psi_1'}{\partial \xi_1}\Delta_2 \psi_2' + \frac{\partial \psi_2'}{\partial \xi_2}\Delta_1 \psi_1'\right)_{\xi_1=\xi_2=1} = k\left(\Delta_2 \frac{\partial B_\psi^\psi}{\partial \xi_1} + \Delta_1 \frac{\partial B_\psi^\psi}{\partial \xi_2}\right)_{\xi_1=\xi_2=1}.$$

*Note: Figure translations are in progress. See original paper for figures.*

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