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Spectral Representations of the Mass and Polarization Operators at Arbitrary Temperature

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§ 1. **Mass operator.** The spectral representation of the mass operator at temperature $T = 0^\circ$ was obtained in works ([1,2]). The present note contains a generalization of the results of ([2]) to the case of arbitrary temperature. For this purpose, apparently, the apparatus of causal Green' s functions used in ([2]) is inconvenient. It is more natural to use advanced and retarded functions, which, as is known ([3]), at any temperature possess simple analytic properties.

We shall consider a spatially homogeneous system. Denote by $G(p, E)$ the Fourier image of the one-fermion retarded or advanced (depending on the sign of $\operatorname{Im} E$) Green' s function. Let G_0 denote the same function in the absence of interaction. Then, by definition ([4]), the mass operator is given by the equality (here and below we omit the argument p)

$$M(E) = G^{-1}(E) - G_0^{-1}(E). \tag{1}$$

For $T = 0$ this, of course, coincides with the definition of M through the causal Green' s function.

The subsequent arguments can be carried out literally in the same way as in ([2]). As is known ([3]), in the complex E-plane ($\hbar = 1$)

$$G_0^{-1}(E) = 2\pi(W - E); \tag{2}$$

$$G(E) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dE' \, \frac{I_+(E')}{E' - E}, \tag{3}$$

where $W(p)$ is the unperturbed energy of one particle, and the spectral function $I_+(E')$ for $E' \in \mathrm{Re}$ has the following properties:

$$I_+(E') \in \mathrm{Re}, \qquad I_+(E') \geqslant 0, \qquad \int_{-\infty}^{+\infty} dE' \, I_+(E') = 1. \tag{4}$$

We shall consider only the region of finite $W(p)$. Then, according to (1)–(4):

$$\frac{M(E)}{|E|} \xrightarrow[|E|\to\infty]{} 0, \qquad \frac{\operatorname{Im} M(E)}{\operatorname{Re} M(E)} \xrightarrow[|E|\to\infty]{} 0. \tag{5}$$

Further, as in ([2]), it is easy to see that

$$\operatorname{sign} \operatorname{Im} M(E) = -\operatorname{sign} \operatorname{Im} E, \qquad M^*(E) = M(E^*). \tag{6}$$

Finally, from formula (3) it is clear that the zeros of the function $G(E)$ (i.e., the poles of $M(E)$), generally speaking, are simple and may lie only on the real axis; their position is determined by the formulas

$$I_+(E) = 0; \tag{7a}$$

$$\int_{-\infty}^{+\infty} dE' \, \frac{I_+(E')}{E' - E} = 0. \tag{7b}$$

Equations (7a, b) can, generally speaking, be satisfied only for a discrete set of values $E = E_i$, $i = 1, 2, \ldots$. It is easy to see further that, by virtue of (6),

$$a_i \equiv \operatorname{res}[M(E), E_i] \in \operatorname{Re}, \qquad a_i \geq 0. \tag{8}$$

At points different from E_i, the function $M(E)$ is analytic in the upper or lower half-plane, respectively, in accordance with the known analyticity properties of G.

The spectral representation for M is now obtained directly. Let us first consider the special case when

$$\lim_{|E|\to\infty} M(E) = c < \infty. \tag{9}$$

This, apparently, is the situation in a number of problems of actual interest. Let us form the function

$$\mathfrak{M}(E) = M(E) - c - \sum_i \frac{a_i}{E - E_i}$$

and apply Cauchy' s theorem to it, integrating over a contour composed of the real axis and a large semicircle in the upper (for $\operatorname{Im} E > 0$) or lower (for $\operatorname{Im} E < 0$) half-plane. We obtain

$$M(E) = c + \sum_i \frac{a_i}{E - E_i} + \frac{1}{\pi} \int_{-\infty}^{+\infty} dE' \, \frac{\rho(E')}{E - E'}, \tag{10}$$

where $c \in \mathrm{Re}$, $\rho \in \mathrm{Re}$, $\rho \geq 0$. Here, evidently,

$$\rho(E') = -\lim_{\varepsilon \to 0} \mathrm{Im}\, M(E' + i\varepsilon) = \lim_{\varepsilon \to 0} \mathrm{Im}\, M(E' - i\varepsilon), \qquad \varepsilon > 0, \quad E' \in \mathrm{Re}\,. \tag{11}$$

For the function $\mathfrak{M}(E)$ the usual dispersion relations hold. Of course, the quantities a_i, E_i, c, ρ generally depend on p, as well as on the temperature, etc.

Thus formula (10) (like (3)) defines two, generally speaking, different analytic functions, and one should speak of two mass operators corresponding, respectively, to the retarded and advanced Green' s functions. The exception is the special case considered in ([2]), when the spectral function $\Gamma(E)$ (and together with it also $\rho(E)$) vanishes identically on some interval of the real axis and an analytic continuation of $G(E)$ from the upper half-plane into the lower is possible.

In the case where condition (9) is not satisfied, one must resort to the standard subtraction procedure ([5]). Putting

$$M(E) = M_{\mathrm{reg}}(E) + \sum_i \frac{a_i}{E - E_i} \tag{12}$$

and applying Cauchy' s theorem to the function

$$m(E) = \frac{M_{\mathrm{reg}}(E) - M_{\mathrm{reg}}(E_0)}{E - E_0}, \tag{13}$$

where E_0 is an arbitrary real constant, we obtain for M_{reg} the usual spectral representation

$$M_{\mathrm{reg}}(E) = \frac{E - E_0}{\pi} \int_{-\infty}^{+\infty} dE' \, \frac{\rho(E')}{(E' - E_0)(E' - E)} + M_{\mathrm{reg}}(E_0), \tag{14}$$

where $\rho(E')$ is given, as before, by formulas of the type (11).

§ 2. **Polarization operator.** Let $D(p, E)$ be the photon or phonon retarded (advanced) Green function (for brevity we omit tensor and other indices), and let D_0 be the same function in the absence of interaction. The polarization operator is defined by the equality [3]

$$P(E) = D_0^{-1}(E) - D^{-1}(E), \tag{15}$$

where

$$D^{-1}_{0,\text{phot}} = (2\pi)^4 \left(p^2 - \frac{E^2}{c^2} \right); \tag{16a}$$

$$D^{-1}_{0,\text{phon}} = -(2\pi)^4 \frac{\omega^2(p) - E^2}{p^2}. \tag{16b}$$

Here c is the speed of light in the medium (assumed nonabsorbing), $\omega(p)$ is the frequency of phonons in the absence of their interaction with electrons, etc.; formula (16b) is written in the approximation usual for homeopolar crystals (isotropic deformation potential); the generalization to more complicated cases presents no difficulty, since for us only the presence of the term E^2 in the numerator is essential.

Further, in the complex E-plane,

$$D(E) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dE' \, \frac{I(E')(1 - e^{-\beta E'})}{E' - E}, \qquad \beta = \frac{1}{\varkappa T}. \tag{17}$$

Using the commutation rules for the operators of the electromagnetic field, it is easy to verify that

$$\int_{-\infty}^{+\infty} dE' \, I(E')(1 - e^{-\beta E'}) = 0; \tag{18a}$$

$$\int_{-\infty}^{+\infty} dE' \, I(E')(1 - e^{-\beta E'}) E' = -\frac{c^2}{(2\pi)^3}. \tag{18b}$$

Analogous formulas are also valid for the phonon field. Accordingly, for finite p^2, $\omega^2(p)$,

$$\left. \frac{P(E)}{E^2} \right|_{|E| \to \infty} \to 0. \tag{19}$$

For the investigation of the zeros of the function $D(E)$ (i.e., possible poles of the polarization operator), it is convenient to transform equality (17) somewhat. Namely, using the easily proved (cf. [6]) equality

$$I(p, -E') = I(p, E') e^{-\beta E'}, \qquad E' \in \text{Re}, \tag{20}$$

we obtain

$$D(E) = \frac{1}{\pi}\int_0^\infty dE' \, \frac{E' I(E')(1-e^{-\beta E'})}{{E'}^2 - E^2} = \frac{1}{2\pi}\int_0^\infty dx \, \frac{I\_(x)}{x - E^2}, \qquad (21)$$

where

$$I\_(x) = I(E')(1-e^{-\beta E'}) \quad \text{for } E' = \sqrt{x},$$

the arithmetic value of the root being meant. Obviously, $I\_(x) \geq 0$.

According to (21), the zeros of $D(E)$, if they exist at all, lie on the real axis at points $E = E_i$ satisfying the equalities

$$I(E)(1-e^{-\beta E}) = 0; \qquad (22a)$$

$$\int_0^\infty dx \, \frac{I\_(x)}{x - E^2} = 0. \qquad (22b)$$

From this it is seen, in particular, that a first-order pole at infinity—

···is, the condition (19) that is not excluded is in fact impossible. Further, as in the case of the mass operator, multiple zeros of $D(E)$ are, generally speaking, impossible—their existence would lead to one more equality, generally speaking incompatible with (22a, b). The obvious exception is the case $E = 0$: if the polarization operator has a pole, then this pole is double. Since the left-hand side of (22a) contains an odd function, while the left-hand side of (22b) contains an even function of E, one may assert that the poles of $P(E)$ are located at points of the real axis symmetric with respect to the origin.* According to (17) and (15), $P^*(E) = P(E^*)$. Consequently, the residues of P are real. Their signs are easily established by noting that, by virtue of (21) and (15),

$$\operatorname{sign} \operatorname{Im} P(E) = -\operatorname{sign}(\operatorname{Re} E \cdot \operatorname{Im} E). \qquad (23)$$

Thus, the sign of the residue of P at the point E_i coincides with the sign of E_i; the absolute values of the residues at the points E_i and $-E_i$ are the same by virtue of the evenness of $\operatorname{Re} P(E)$ for real E. For $E_i = 0$ the residue, as one can verify, is positive.

The spectral representation for $P(E)$ can now be written in complete analogy with (10) and (14), and with the same conclusions. Let us note that, owing to gradient invariance, the residues $a_i(p)$ in the electromagnetic case vanish at $p = 0$ (in a system with an inversion center, no more slowly than quadratically).

The presence of poles in the polarization operator of the electromagnetic problem obviously means the appearance of infinite peaks in the real part of the electrical conductivity at frequencies equal to E_i (excluding the case $E_i = 0$). The dispersion law (both for photons and for phonons) may then take a rather unusual form. Thus, for $E_i = 0$ and small p, one obtains $E \sim p^{2/3}$.

Let us note that, in the nonrelativistic formulation of the problem, the function $D_{0,\text{phot}}$ does not depend on E at all. It might seem that this leads to a double pole of P at infinity. Such a conclusion, however, would be inconsistent, since the limiting transition $c \to \infty$ leads to divergence of the integral (18b).

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* One may, in general, restrict oneself to the values $E \geq 0$ and consider P as a function of E^2. This, however, is not always convenient.

Note: Figure translations are in progress. See original paper for figures.