MATHEMATICS

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1962

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Abstract

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MATHEMATICS

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ON SOME DIFFERENTIAL EQUATIONS ARISING IN THE STUDY OF OSCILLATIONS WITH SMALL RANDOM PERTURBATIONS

(Presented by Academician I. G. Petrovskii on 18 IX 1961)

1. An oscillatory system subject to random perturbations is, under certain assumptions on the nature of the perturbations, a Markov random process of diffusion type. It is of interest to study the limiting distribution of various characteristics associated with this process when the random perturbation tends to zero. This problem is connected with the study of solutions of a certain class of elliptic and parabolic equations of second order with a small parameter ε multiplying the highest derivatives. This class of equations is characterized by the fact that for $\varepsilon = 0$ (absence of perturbations) it describes motion along closed curves without absorption (the case with absorption is treated in (^{1})). If this motion is nonrandom, then (considering for the time being the elliptic case) by a suitable choice of the coordinate system one may restrict oneself to studying the behavior, as $\varepsilon \to 0$, of the solution u_ε of the equation

$$\left(L_1(x) + V(x) + \frac{1}{\varepsilon}\frac{\partial}{\partial x_2}\right) u = \sum_{i,j=1}^{2} a_{ij}(x_1, x_2)\frac{\partial^2 u}{\partial x_i \partial x_j} +$$

$$+ \sum_{i=1}^{2} b_i(x_1, x_2)\frac{\partial u}{\partial x_i} + V(x_1, x_2)u + \frac{1}{\varepsilon}\frac{\partial u}{\partial x_2} = -F(x_1, x_2) \tag{1}$$

in the circular annulus K ($K = \{(x_1, x_2) : r_1 < x_1 < r_2\}$, where (x_1, x_2) and $(x_1, x_2 + 1)$ represent one and the same point), satisfying the conditions

$$u_\varepsilon(r_i, x_2) = f_i(x_2). \tag{1'}$$

We shall carry out such an investigation for (1), (1′) and for the more general equation (2) (i.e., for the case where the "principal" motion along closed trajectories is also random), and we shall also consider the parabolic equation.

2. Suppose that the operator $L_1(x)$ is elliptic everywhere in $\overline{K}$ ($\overline{K} = K \cup \Gamma$; $\Gamma = \Gamma_1 \cup \Gamma_2$; $\Gamma_i = \{x : x_1 = r_i\}$). Put

$$L_2(x) = A(x)\frac{\partial^2}{\partial x_2^2} + B(x)\frac{\partial}{\partial x_2} \quad (A > a_0 > 0 \text{ in } K),$$

$$L_2^* u = \frac{\partial^2}{\partial x_2^2}(A(x)\cdot u) - \frac{\partial}{\partial x_2}(B(x)\cdot u)$$

and assume that the coefficients of the operators L_1 and L_2 are twice continuously differentiable in K, with the first derivatives continuous in $\overline{K}$. Let V and F be twice continuously differentiable complex-valued functions, with $\operatorname{Re} V(x) = V_1(x) \leq 0$, and let f_i $(i = 1, 2)$ be continuous complex-valued functions on Γ_i. Denote by $\mu(x_1^0, x_2) =$ the density of the invariant measure of the Markov random process on the circle of the surface $l_{x_1^0} = \{x : x_1 = x_1^0\}$, corresponding to the operator L_2 (μ is the unique solution of the equation $L_2^*\mu = 0$ on $l_{x_1^0}$, normalized by the condition $\int_0^1 \mu \, dx_2 = 1$; in particular, $\mu = 1$ if $L_2 = L_2^*$). For any integrable function $g(x)$ put

$$\tilde{g}(x_1) = \int_0^1 g(x_1, x_2)\mu(x_1, x_2)\, dx_2.$$

Theorem 1. *The solution $u_\varepsilon(x)$ of the equation*

$$\left(L_1 + V + \frac{1}{\varepsilon}L_2\right) u = -F \tag{2}$$

in K, satisfying condition $(1')$, converges as $\varepsilon \to 0$ to the solution $u_0(x_1)$ of the equation

$$[\widetilde{L}_1(x_1) + \widetilde{V}(x_1)]u_0 = -\widetilde{F}(x_1), \tag{3}$$

satisfying the conditions

$$u_0(r_i) = \frac{\displaystyle\int_0^1 a_{11}(r_i, x_2)\mu(r_i, x_2) f_i(x_2)\, dx_2}{\displaystyle\int_0^1 a_{11}(r_i, x_2)\mu(r_i, x_2)\, dx_2} = \hat{f}_i \quad (i = 1, 2). \tag{3'}$$

Moreover, uniformly in any closed subdomain of K, the relation

$$u_\varepsilon(x) - u_0(x_1) = O(\sqrt{\varepsilon})$$

* The equation (1) with a complex function V is satisfied, for example, by characteristic functions of certain f

holds. In the case $f_i(x_2) = \text{const}$*, this estimate can be improved:* $u_\varepsilon(x) - u_0(x_1) = O(\varepsilon)$ *uniformly in* K*.*

Theorem 1 is also valid for the case where the domain under consideration is a disk. (One of the boundary conditions ($3'$) must then be replaced by the condition of boundedness at zero of the solution of equation (3).) In addition, Theorem 1 remains valid also in the case $A \equiv 0$, $B \neq 0$. Some probabilistic consequences of Theorem 1 with $A \equiv 0$ are given in (^{3}).

We now somewhat modify the restrictions adopted above, allowing the coefficients of the operator L, V, and F also to depend on t ($0 \leq t \leq T$), in such a way that $L_1 = L_1(x,t)$, $V(x,t)$, and $F(x,t)$ are twice differentiable with respect to all their arguments in the domain $K \times (0,T)$.

Theorem 2. *The solution* $u_\varepsilon(x,t)$ *of the equation*

$$\frac{\partial u}{\partial t} = L_1(x,t)u + V(x,t)u + \frac{1}{\varepsilon}L_2(x)u + F(x,t)$$

in the domain $K \times (0,T)$*, satisfying the conditions*

$$u_\varepsilon(x_1,x_2,0) = f(x_1,x_2), \qquad u_\varepsilon(r_i,x_2,t) = f_i(x_2,t) \quad (i=1,2),$$

as $\varepsilon \to 0$ *converges to the solution* $u_0(x_1,t)$ *of the equation*

$$\partial u/\partial t = [\widetilde{L}_1(x_1,t) + \widetilde{V}(x_1,t)]u + \widetilde{F}(x_1,t),$$

satisfying the conditions

$$u_0(x_1,0) = \tilde{f}(x_1), \qquad u_0(r_i,t) = \tilde{f}_i(t) \quad (i=1,2).$$

Moreover, $u_\varepsilon(x,t) - u_0(x_1,t) = O(\sqrt{\varepsilon})$ *uniformly in any closed domain contained in* $K \times (0,T)$*.*

It is interesting to note that $\lim_{\varepsilon\to 0} u_\varepsilon(x,t)$, generally speaking, does not exist if $A \equiv 0$. We also note that a result analogous to Theorem 2 is valid for the solution of the Cauchy problem as well.

3. The proof of Theorem 1 uses the following lemmas.

Lemma 1. *Under the assumptions of Theorem 1, the solution* $Z(x)$ *of the equation*

$$L_1 Z + VZ = -F$$

in the domain K*, satisfying the conditions* $Z(r_i,x_2) = f_i(x_2)$*, admits the estimate*

$$|Z| \leq \max_{x_2,\ i=1,2} |f_i| + C \min_{i=1,2} \int_{r_1}^{r_2} |r_i - x_1| \max_{x_2} |F(x_1,x_2)|\, dx_1.$$

Here the constant C *depends only on the upper and lower bounds of the coefficients* a_{11} *and* b_1 *in the domain* K*.*

Lemma 2. Let x_2 be the coordinate of a point on the circle l (considered modulo 1); $A_1(x_2) > 0$, $B_1(x_2)$ are twice continuously differentiable functions on l. The equation $Mu = A_1 d^2u/dx_2^2 + B_1 du/dx_2 = \Phi(x_2)$ has a solution on l if and only if

$$\int_0^1 \Phi(y)\mu(y)\,dy = 0$$

($\mu(x_2)$ is a solution of the equation $M^*\mu = 0$, satisfying the condition

$$\int_0^1 \mu(y)\,dy = 1$$

).

Lemma 3. The solution $u(x_1, x_2)$ of the elliptic equation $\partial^2 u/\partial x_1^2 + Mu = 0$ in the domain $G = \{x_1 > 0\} \times l$, satisfying the condition $u_0(0, x_2) = f(x_2)$, admits the estimates:

$$\left| u(x_1, x_2) - \int_0^1 f(y)\mu(y)\,dy \right| \le c_1 e^{-c_2 x_1}; \qquad |u'_{x_i}| \le c_3 e^{-c_2 x_1}; \qquad |u''_{x_i,x_j}| < c_3 e^{-c_2 x_1}$$

(here and below the c_i are certain positive constants).

Let $u_\varepsilon(x)$ be the solution of equation (2), (1′), and let $v_\varepsilon^{(1)}(x)$ be the solution of equation (2), satisfying the conditions $v_\varepsilon^{(1)}(r_i, x_2) = \hat{f}_i$ $(i = 1, 2)$, $v_\varepsilon^{(2)}(x) = u_\varepsilon(x) - v_\varepsilon^{(1)}(x)$. We shall prove the following estimates:

$$v_\varepsilon^{(1)}(x) - u_0(x_1) = O(\varepsilon) \quad \text{uniformly in } K; \tag{4}$$

$$v_\varepsilon^{(2)}(x) = O(\sqrt{\varepsilon}) \quad \text{uniformly in any closed subdomain of } K. \tag{5}$$

Set $\Phi(x) = L_1(x) \cdot u_0 + F(x) + V(x) \cdot u_0$. Since, by virtue of (3), $\widetilde{\Phi}(x_1) = 0$, applying Lemma 2 one may assert that the equation $L_2 w = \Phi$ has a solution on each circle l_{x_1}, $r_1 \le x_1 \le r_2$, where the variable x_1 enters this equation as a parameter. It is not difficult to show that w may be chosen twice continuously differentiable with respect to x_1, since A, B, and Φ possess this property. Now considering the function $\Psi_\varepsilon(x) = v_\varepsilon^{(1)}(x) - u_0(x_1) + \varepsilon w(x)$, it is easy to see that $\Psi_\varepsilon(r_i, x_2) = \varepsilon w(r_i, x_2) = O(\varepsilon)$ and

$$\left(L_1 + V + \frac{1}{\varepsilon} L_2 \right) \Psi_\varepsilon = \varepsilon \left[L_1 w + V w \right] = O(\varepsilon).$$

Hence, applying Lemma 1, we obtain (4). To prove (5) we construct auxiliary functions $g_i(x_1, x_2, \varepsilon)$ $(i = 1, 2)$ so that the function g_i has the character of a boundary layer near Γ_i. We shall start here from the function $Z_i(x_1, x_2)$, defined in G as the solution of the equation

$$L_2(r_i, x_2)Z + a_{11}(r_i, x_2)\frac{\partial^2 Z}{\partial x_1^2} = 0; \tag{6}$$

$$Z(0, x_2) = f_i(x_2) - \hat{f}_i. \tag{7}$$

Application of Lemma 4 to (6), (7) makes it possible to establish that

$$|Z_i| < c_1 e^{-c_2 x_1}; \qquad \left|\frac{\partial Z_i}{\partial x_j}\right| < c_3 e^{-c_2 x_1}; \qquad \left|\frac{\partial^2 Z_i}{\partial x_j \partial x_k}\right| < c_3 e^{-c_2 x_1}.$$

Putting now

$$g_i(x_1, x_2, \varepsilon) = Z_i\left(\frac{|x_1 - r_i|}{\sqrt{\varepsilon}}, x_2\right),$$

we therefore obtain:

$$g_i,\ \frac{\partial g_i}{\partial x_2},\ \frac{\partial^2 g_i}{\partial x_2^2} = O\left[\exp\left(-\frac{c_2|x_1 - r_i|}{\sqrt{\varepsilon}}\right)\right];$$

$$\frac{\partial g_i}{\partial x_1},\ \frac{\partial^2 g_i}{\partial x_1 \partial x_2} = O\left[\frac{1}{\sqrt{\varepsilon}}\exp\left(\frac{c_2|x_1 - r_i|}{\sqrt{\varepsilon}}\right)\right];$$

$$\frac{\partial^2 g_i}{\partial x_1^2} = O\left[\frac{1}{\varepsilon}\exp\left(-c_2\frac{|x_1 - r_i|}{\sqrt{\varepsilon}}\right)\right].$$

Next, using (6) and these estimates, we have:

$$\begin{aligned}\left(L_1 + V + \frac{1}{\varepsilon}L_2\right)g_i(x, \varepsilon) = {}& [a_{11}(x_1, x_2) - a_{11}(r_i, x_2)]\frac{\partial^2 g_i}{\partial x_1^2} + \\ &+ \frac{1}{\varepsilon}[L_2(x_1, x_2) - L_2(r_i, x_2)]\,g_i + O\left[\frac{1}{\sqrt{\varepsilon}}\exp\left(-\frac{c_2|x_1 - r_i|}{\sqrt{\varepsilon}}\right)\right] \\ &= O\left[\frac{|x_1 - r_i|}{\varepsilon}\exp\left(-\frac{c_2|x_1 - r_i|}{\sqrt{\varepsilon}}\right)\right] + O\left[\frac{1}{\sqrt{\varepsilon}}\exp\left(-\frac{c_2|x_1 - r_i|}{\sqrt{\varepsilon}}\right)\right].\end{aligned}$$

Applying now Lemma 1 to the function

$$v_\varepsilon^{(2)}(x) - \sum_{i=1}^{2} g_i(x,\varepsilon),$$

it is easy to verify that

$$v_\varepsilon^{(2)}(x) - \sum_{i=1}^{2} g_i(x,\varepsilon) = O(\sqrt{\varepsilon})$$

uniformly in K, whence (5) follows. Theorem 1 follows from (4) and (5).

4. The proof of Theorem 1 in the case $A \equiv 0,\ B \neq 0$ is analogous. (Instead of Lemma 3 one must use a similar assertion concerning the solution of the equation $\varepsilon^2 u/\partial x_1^2 + \partial u/\partial x_2 = 0$ in G.) The proof of Theorem 2 is constructed according to the same plan. It is only necessary additionally to construct a boundary layer near the plane $t = 0$. For its construction the following is used:

Lemma 4. *The solution of the equation* $\partial u/\partial t = Mu$ *in the domain* $\{t > 0\} \times l$, *satisfying the condition* $u(0, x_2) = f(x_2)$, *admits the estimate*

$$\left| u(t, x_2) - \int_0^1 f(y)\mu(y)\, dy \right| < c_1 e^{-c_2 t}.$$

For the proof of Lemma 1 an auxiliary function (a barrier) depending only on x_1 is constructed. Lemmas 2 and 4 follow, for example, from more general theorems reported by M. I. Freidlin at a conference on probability theory in Vilnius (1960). The proof of Lemma 3 uses the probabilistic representation of the solution of the equation $\partial^2 u/\partial x_1^2 + Mu = 0$ and Lemma 4.

The method of proof of Theorem 1 is also generalized to the case of a larger number of dimensions*.

I note in conclusion that the result obtained in Theorem 1 for the case $A \equiv 0,\ f_i = \text{const}$ was, in its general features, predicted earlier by A. N. Kolmogorov during discussion of the works of O. A. Oleinik and her students.

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Received
13 IX 1961

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* **Note added in proof.** A. N. Kolmogorov, having read the manuscript of this paper, expressed the hypothesis that the next term of the asymptotic expansion of $u_\varepsilon(x)$ in powers of $\sqrt{\varepsilon}$ does not depend on x_2. Indeed, substituting

$$v_\varepsilon^{(2)}(x) = z_{11}\left(\frac{x_1 - r_1}{\sqrt{\varepsilon}}, x_2\right) + z_{12}\left(\frac{x_1 - r_2}{\sqrt{\varepsilon}}, x_2\right) +$$

$$+\sqrt{\varepsilon}\left[z_{21}\left(\frac{x_1 - r_1}{\sqrt{\varepsilon}}, x_2\right) + z_{22}\left(\frac{x_1 - r_2}{\sqrt{\varepsilon}}, x_2\right)\right] + \cdots$$

in (2) and using the estimates given above, it is not difficult to show that

$$u_\varepsilon(x) = u_0(x_1) + \sqrt{\varepsilon}\,(c_1 + c_2) + O(\varepsilon)$$

in any closed subdomain of K. Here the constant c_i is determined by the values of the coefficients and of their derivatives with respect to x_1 on Γ_i. One can also obtain the subsequent terms of the asymptotic expansion, adapting to the present case the method developed in (2).

Note: Figure translations are in progress. See original paper for figures.

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