ON THE SET SWEPT OUT BY A LINE SEGMENT

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Abstract

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ON THE SET SWEPT OUT BY A LINE SEGMENT

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Let $l(x)$ be a line in the plane x, y, passing through the point x of the axis of abscissas and forming an angle $\alpha(x) \neq 0$ with the positive direction of the x-axis. Denote by $\Delta(x)$ the segment of the line $l(x)$ between the axis of abscissas and the line $y = 1$. Suppose the line $l(x)$ is defined for all x in the segment $[0, 1]$. If $\alpha(x)$ is a continuous function, then the segment $\Delta(x)$, as the point x moves along the segment $[0, 1]$, sweeps out an area not less than $1/2$.

What is the situation if $\alpha(x)$ is discontinuous?

In this note it will be shown that there exists such a measurable function $\alpha(x)$ that $\Delta(x)$ sweeps out zero area. Moreover, the function $\alpha(x)$ can be constructed so that the endpoint of the segment $\Delta(x)$, moving along the line $y = 1$, traverses the segment $0 \leq x \leq 1$, hitting each point exactly once.

Let a parallelogram π be given. Fix two of its opposite sides (we shall call them bases).

We shall call an a-system a finite system of parallelograms located in π, such that two opposite sides of each parallelogram lie on the bases of π, the bases of π being completely covered by these sides, while the sides of different parallelograms lying on a base of π may intersect only at their endpoints. The set-theoretic sum of the parallelograms of a given a-system A will be denoted by $|A|$.

Suppose that in some fixed parallelogram π an a-system A is given, consisting of parallelograms $\pi_1, \dots, \pi_n$, and suppose that in each parallelogram π_i ($i = 1, \dots, n$) an a-system A_i is given (where as bases of π_i the sides lying on the bases of π are taken). Then all the parallelograms of all the systems A_i ($i = 1, \dots, n$) form an a-system $\widetilde{A}$ in π. We shall say that $\widetilde{A}$ is embedded in A. If $\widetilde{A}$ is embedded in A, then, obviously, $|\widetilde{A}| \subset |A|$ and $\operatorname{mes}|\widetilde{A}| \leq \operatorname{mes}|A|$.

Take a sequence of a-systems $A^1, A^2, \dots, A^n, \dots$ in π such that A^n is embedded in A^{n-1} ($n = 2, 3, \dots$). Then $|A^1|, |A^2|, \dots, |A^n|, \dots$ are nonempty closed sets, and moreover $|A^{n-1}| \supset |A^n|$ ($n = 2, 3, \dots$). Hence their intersection

$$B = \bigcap_{n=1}^{\infty} |A^n|$$

Fig. 1

Figure 1: Fig. 1

is a nonempty closed set, and

$$\operatorname{mes} B = \lim_{n\to\infty} \operatorname{mes} |A^n|.$$

Such sets (intersections of sequences of embedded a-systems) will for brevity be called b-sets.

From the definition of a b-set it follows that for any b-set B and for any $\varepsilon > 0$ one can find an a-system A such that $|A| \supset B$ and $\operatorname{mes} |A| < \operatorname{mes} B + \varepsilon$.

Let $\inf \operatorname{mes} B = \beta$, where the greatest lower bound is taken over all b-sets $B \subset \pi$. We shall prove that the greatest lower bound is attained: there exists a b-set $B_0 \subset \pi$ whose measure is equal to β.

Let an a-system A be given in a parallelogram π, and let the parallelogram π' be obtained from π by an affine transformation α. Under the transformation α, the a-system A passes into some a-system A' in π'. We shall say that A' in π' is induced by the a-system A in π. Note that the ratios $\operatorname{mes} |A| / \operatorname{mes} \pi$ and $\operatorname{mes} |A'| / \operatorname{mes} \pi'$ are equal.

Let an a-system $\widetilde{A}$ in π be embedded in an a-system A in π, and suppose that all A_i in π_i (see the definition of embedded a-systems) are induced by one and the same a-system A^* in π, for all $i = 1, 2, \dots, n$. For this case we introduce the special notation

$$\widetilde{A} = A \cdot A^*.$$

If $\widetilde{A} = A \cdot A^*$, then $\operatorname{mes} |\widetilde{A}| \leqslant \operatorname{mes} |A^*|$. Indeed,

$$\operatorname{mes} |\widetilde{A}| \leqslant \sum_i \operatorname{mes} |A_i| =$$

$$= \sum_i \operatorname{mes} |A^*| \frac{\operatorname{mes} \pi_i}{\operatorname{mes} \pi} =$$

$$= \operatorname{mes} |A^*| \sum_i \frac{\operatorname{mes} \pi_i}{\operatorname{mes} \pi} = \operatorname{mes} |A^*|.$$

Fig. 1

Now choose a sequence of b-sets $B_n \subset \pi$ such that

$$\lim_{n\to\infty} \operatorname{mes} B_n = \beta.$$

Let, further, for each $n = 1, 2, \dots$ the a-system A_n in π be such that

$$\operatorname{mes} |A_n| < \operatorname{mes} B_n + \frac{1}{2^n}.$$

Since for every a-system A the set $|A|$ is, obviously, a b-set, we have $\operatorname{mes}|A_n| \geqslant \beta$. Consequently,

$$\lim_{n\to\infty} \operatorname{mes}|A_n| = \beta.$$

We now construct a new sequence of A-systems:

$$\widetilde{A}_1 = A_1, \qquad \widetilde{A}_n = \widetilde{A}_{n-1} \cdot A_n \quad (n = 2, 3, ...).$$

By what was proved above,

$$\operatorname{mes}|\widetilde{A}_n| \leqslant \operatorname{mes}|A_n|.$$

Consequently,

$$\lim_{n\to\infty} \operatorname{mes}|\widetilde{A}_n| = \beta.$$

Since $\{\widetilde{A}_n\}$ is a sequence of embedded a-systems, it follows that

$$B_0 = \bigcap_{n=1}^{\infty} |\widetilde{A}_n|$$

is a b-set, and $\operatorname{mes} B_0 = \beta$. We now show that $\beta = 0$. Suppose the contrary: $\beta > 0$. We shall be able to construct a b-set B^*, whose measure is less than that of B_0, and thus arrive at a contradiction.

Thus, let $\beta > 0$. Then in B_0 there is a density point P which does not lie on the base of π. Consider the a-system A^* in π shown in Fig. 1. A^* consists of the four parallelograms $\pi_1, \pi_2, \pi_3, \pi_4$. The parallelograms π_1 and π_2 are chosen so that their lateral sides, respectively, intersect on the straight line parallel to the base of π and passing through the point P. Then, under the affine transformations carrying π respectively into π_1 and into π_2, the images of the point P coincide.

Let, under the affine transformation α_i carrying π into π_i $(i = 1, 2, 3, 4)$, the b-set B_0 pass into the set B_{0i}, and let

$$B^* = \bigcup_{i=1}^{4} B_{0i}.$$

The set B^* is a b-set, and $\operatorname{mes} B^* < \operatorname{mes} B_0$. The contradiction obtained shows that $\beta = 0$.

Thus the existence has been proved of a closed set $B_0 \subset \pi$ of measure zero, containing both bases of π, and such that together with each of its points there belongs to it a rectilinear segment from one base

to another containing this point. At the same time, the existence of a-systems of arbitrarily small area has been proved.

Let π be the square $0 \leq x \leq 1,\ 0 \leq y \leq 1$. The set B_0 does not yet give the function we need, since the segments joining in it the points of the bases of the square π do not establish a one-to-one correspondence. But this is easy to correct.

Recall that

$$B_0 = \bigcap_{n=1}^{\infty} \widetilde{A}_n.$$

For the a-system $\widetilde{A}_n$, denote by N_n^1 and N_n^2 the sets of vertices of the parallelograms entering into $\widetilde{A}_n$, with N_n^1 being the set of vertices situated on the lower base of π, and N_n^2 the set of vertices situated on the upper base of π. Let

$$N_1 = \bigcup_{n=1}^{\infty} N_n^1 \quad \text{and} \quad N_2 = \bigcup_{n=1}^{\infty} N_n^2.$$

Each of the sets N_1 and N_2 is countable.

Remove N_1 and N_2, respectively, from the lower and upper bases of π, and denote the remaining sets by D_1 and D_2.

For each point of D_1 there exists a unique segment in B_0 joining this point with some point of D_2, and conversely. Denoting by $\alpha(x)$ the angle of inclination of the segment joining the point x of D_1 with D_2, we note that the function $\alpha(x)$ is continuous on D_1. Put the points of the sets N_1 and N_2 into one-to-one correspondence in some way and join the corresponding points by segments. We obtain a one-to-one mapping of the entire lower side of the square onto the upper side. The function $\alpha(x)$ is thereby supplemented by certain values on the countable set N_1 and will thus be measurable.

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Note: Figure translations are in progress. See original paper for figures.