MATHEMATICS

V. F. VLASOV

1962

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. F. VLASOV

A CONSTRUCTIVE CHARACTERIZATION OF ONE CLASS OF FUNCTIONS

(Presented by Academician S. N. Bernstein on 19 IX 1961)

Let $f(x)$ be a continuous 2π-periodic function, and let $\tilde{f}(x)$ be trigonometrically conjugate to $f(x)$. By a theorem of I. I. Privalov, $f(x)$ and $\tilde{f}(x)$ simultaneously satisfy a Lipschitz condition of order α ($\operatorname{Lip}\alpha$) if $0 < \alpha < 1$, but for $\alpha = 1$ this property of the function fails: there exist functions satisfying the condition $\operatorname{Lip} 1$ whose trigonometric conjugates do not satisfy this condition. The following theorem gives, in terms of interpolation theory, necessary and sufficient conditions for a function and its trigonometric conjugate simultaneously to satisfy the condition $\operatorname{Lip} 1$.

Theorem 1. In order that $f(x)$ and $\tilde{f}(x)$ simultaneously satisfy the condition $\operatorname{Lip} 1$, it is necessary and sufficient that

$$f(x) - \tau_n(x) = O\left(\frac{1}{n}\right) \tag{1}$$

uniformly with respect to x and n, where

$$\tau_n(x) = \tau_n(f;x) = \frac{1}{n^2}\sum_{\nu=0}^{n-1} f(x_\nu)\,\frac{\sin^2\frac{n}{2}(x_\nu - x)}{\sin^2\frac{1}{2}(x_\nu - x)}, \qquad x_\nu = \frac{2\nu\pi}{n}.$$

The trigonometric polynomial $\tau_n(x)$, called the Jackson interpolation polynomial, has, as is known, the property that $\tau_n(x_\nu) = f(x_\nu)$ and $\tau_n'(x_\nu) = 0$ ($\nu = 0, 1, \dots, n-1$).

Proof. Using the well-known method of S. N. Bernstein, it is easy to show that from the condition $f(x) - T_n(x) = O\left(\frac{1}{n}\right)$, where $T_n(x)$ is an arbitrary trigonometric polynomial of order n, it follows that

$$T_n''(x) = O(n) \tag{2}$$

uniformly in x and n. Hence, by virtue of the noted property of the polynomial $\tau_n(x)$ and (1), it follows that

$$\tau_n'(x) = \tau_n'(x) - \tau_n'(x_\nu) = \tau_n''(c)(x - x_\nu) = O(1) \tag{3}$$

uniformly with respect to x and n, where $x_\nu \le x < x_{\nu+1}$, $x_\nu \le c < x_{\nu+1}$. Therefore, whatever $0 < h < 1$ may be, choosing n so that $\frac{1}{h} - 1 \le n < \frac{1}{h}$, we shall have

$$|f(x+h) - f(x)| \le 2 \max_x |f(x) - \tau_n(x)| + |\tau_n(x+h) - \tau_n(x)| < C_1 h,$$

i.e., $f(x)$ satisfies the condition Lip 1.

Next, we introduce into consideration the Fejér sums of the Fourier series of the function $f(x)$:

$$\sigma_n(x) = \sigma_n(f, x) = \frac{1}{2n\pi} \int_0^{2\pi} f(t) \frac{\sin^2 \frac{n}{2}(x-t)}{\sin^2 \frac{1}{2}(x-t)} \, dt$$

and find an estimate for the magnitude of the difference

$$\sigma_n(x) - \tau_n(x) = \frac{1}{2n\pi} \int_0^{2\pi/n} \sum_{\nu=0}^{n-1} \left[f(x_\nu + t) \frac{\sin^2 \frac{n}{2}(x_\nu + t - x)}{\sin^2 \frac{1}{2}(x_\nu + t - x)} - f(x_\nu) \frac{\sin^2 \frac{n}{2}(x_\nu - x)}{\sin^2 \frac{1}{2}(x_\nu - x)} \right] dt. \tag{4}$$

Replacing in this expression $f(x_\nu)$ and $f(x_\nu + t)$ by $\tau_n(x_\nu)$ and $\tau_n(x_\nu + t)$, respectively, and taking (1) into account, we have

$$\sigma_n(x) - \tau_n(x) = \frac{1}{2n\pi} \int_0^{2\pi/n} \sum_{\nu=0}^{n-1} \tau_n(x_\nu) \left[\frac{\sin^2 \frac{n}{2}(x_\nu + t - x)}{\sin^2 \frac{1}{2}(x_\nu + t - x)} - \frac{\sin^2 \frac{n}{2}(x_\nu - x)}{\sin^2 \frac{1}{2}(x_\nu - x)} \right] dt +$$

$$+ \frac{1}{2n\pi} \int_0^{2\pi/n} \sum_{\nu=0}^{n-1} [\tau_n(x_\nu + t) - \tau_n(x_\nu)] \frac{\sin^2 \frac{n}{2}(x_\nu + t - x)}{\sin^2 \frac{1}{2}(x_\nu + t - x)} \, dt + O\left(\frac{1}{n}\right).$$

Using Taylor' s formula and the property of $\tau_n(x)$, we obtain

$$\sigma_n(x)-\tau_n(x) = -\frac{1}{2n\pi}\int_0^{2\pi/n}\sum_{\nu=0}^{n-1} f(x_\nu)\left[t\frac{d}{dx}\frac{\sin^2\frac{n}{2}(x_\nu-x)}{\sin^2\frac{1}{2}(x_\nu-x)}+\right.$$
$$\left.+\frac{t^2}{2}\frac{d^2}{dx^2}\frac{\sin^2\frac{n}{2}(x_\nu-x+\theta)}{\sin^2\frac{1}{2}(x_\nu-x+\theta)}\right]dt+$$
$$+\frac{1}{2n\pi}\int_0^{2\pi/n}\sum_{\nu=0}^{n-1} t\,\tau_n'(x_\nu+\theta_1)\frac{\sin^2\frac{n}{2}(x_\nu+t-x)}{\sin^2\frac{1}{2}(x_\nu+t-x)}\,dt+O\left(\frac{1}{n}\right), \tag{5}$$

where $0 \le \theta, \theta_1 \le 2\pi/n$.

Using relations (2) and (4), from equality (5) we obtain

$$\sigma_n(x)-\tau_n(x) = O\left(\frac{1}{n}\frac{d\tau_n(x)}{dx}+\frac{1}{n^2}\frac{d^2\tau_n(x)}{dx^2}\right)+O\left(\frac{1}{n}\right),$$

i.e.

$$f(x)-\sigma_n(x) = O\left(\frac{1}{n}\right).$$

From this, as Aleksich showed ([1]), it follows that the function trigonometrically conjugate to $f(x)$ exists and satisfies the condition Lip 1.

Now suppose that $f(x)$ and $\tilde{f}(x)$ satisfy the condition Lip 1. In this case (see (1))

$$f(x)-\sigma_n(x) = O\left(\frac{1}{n}\right) \tag{6}$$

uniformly with respect to x and n.

Moreover, from M. Riesz' s inequality (([2], see also ([3]))

$$|T_n'(x)| \le \frac{n}{2}\max_x\left|T_n\left(x+\frac{\pi}{2n}\right)-T_n(x)\right|,$$

where $T_n(x)$ is an arbitrary trigonometric polynomial of order n, it follows that

$$|\sigma_n'(x)| \le \frac{n}{2}\max_x\left|f\left(x+\frac{\pi}{2n}\right)-f(x)\right|+O(1) = O(1) \tag{7}$$

uniformly with respect to x and n.

We shall show that under our assumptions

$$\sigma_n(x) - \tau_n(x) = O\left(\frac{1}{n}\right).$$

Replacing in (4) $f(x_\nu)$ and $f(x_\nu + t)$ by $\sigma_n(x_\nu)$ and $\sigma_n(x_\nu + t)$, respectively, and using (2), (4), (6), and (7), we obtain

$$\begin{aligned}\sigma_n(x) - \tau_n(x) &= \frac{1}{2n\pi}\int_0^{2\pi/n}\sum_{\nu=0}^{n-1} f(x_\nu + t)\left[\frac{\sin^2\frac{n}{2}(x_\nu + t - x)}{\sin^2\frac{1}{2}(x_\nu + t - x)} - \frac{\sin^2\frac{n}{2}(x_\nu - x)}{\sin^2\frac{1}{2}(x_\nu - x)}\right] dt + O\left(\frac{1}{n}\right)\\ &= O\left(\frac{1}{n} + \frac{1}{n}\frac{d\sigma_n(x)}{dx} + \frac{1}{n^2}\frac{d^2\sigma_n(x+\theta_2)}{dx^2}\right) = O\left(\frac{1}{n}\right),\end{aligned} \tag{8}$$

where $0 \le \theta_2 \le 2\pi/n$.

Equalities (6) and (8) prove the necessity in Theorem 1.

In conclusion I consider it my duty to express my deep gratitude to A. F. Timan for formulating the problem and for his constant attention.

Dnepropetrovsk State University
named for the 300th anniversary of the reunification of Ukraine with Russia

Received
1 IX 1961

CITED LITERATURE

[1] G. Alxits, *Matematikai es Fizikai Lapok*, **48**, 410 (1941); *Acta Math. Hung.*, **3**, 29 (1952). [2] M. Riesz, *Jahresber. Deutsch. Math. Ver.*, **23** (1914). [3] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, 1960.

Note: Figure translations are in progress. See original paper for figures.