On the Normability of a Boolean Algebra

D. A. Vladimirov

1962

SovietRxiv

Abstract

Full Text

On the Normability of a Boolean Algebra

D. A. Vladimirov

(Presented by Academician V. I. Smirnov, December 18, 1961)

In the present note we establish some conditions under which a Boolean algebra turns out to be normable, i.e., possesses an essentially positive countably additive measure. If the elements of the algebra are interpreted as events, then this concerns the possibility of assigning to these events probabilities with the usual properties. It is well known that there exist non-normable complete Boolean algebras. Various conditions of normability are given in the papers ([1,2]).

Our approach will consist in studying the totality of automorphisms of a given Boolean algebra. We shall determine what properties of this totality imply the existence of a measure.

In what follows, X denotes a complete Boolean algebra. $0, 1$ are the zero and unit of the algebra, and Cx is the Boolean complement of the element x. The symbol $x \vee y$ denotes the supremum of the elements x and y, and the symbol $x \wedge y$ their infimum. For an arbitrary set of elements $\{x_\alpha\}$, the symbols $\bigvee_\alpha x_\alpha$ and $\bigwedge_\alpha x_\alpha$ are used, respectively. If $x \wedge y = 0$, then x and y are said to be disjoint. In this case their supremum is called their sum and is denoted by $x + y$. Similarly, one writes $\bigvee_\alpha x_\alpha = \sum_\alpha x_\alpha$ if the x_α are pairwise disjoint. If $x \geq y$, then the difference of x and y is the element $x - y = x \wedge Cy$. The sign $\to$ in this note denotes convergence in the topology generated by the ordering ((t)-convergence in the (o)-topology). All necessary information may be found in the monograph ([3]).

By a **measure** on the algebra X we shall always mean a real-valued function μ such that:

1) $\mu(x) > 0$ for $x \neq 0$;
2) if $x \wedge y = 0$, then $\mu(x + y) = \mu(x) + \mu(y)$;
3) if $x_n \geq x_{n+1}$, $n = 1, 2, ...$, and $x_n \to 0$, then $\mu(x_n) \to 0$;
4) $\mu(1) = 1$.

An **automorphism** of the algebra X is a one-to-one mapping A of the algebra onto itself that preserves the ordering (the latter means that the relations $x < y$ and $Ax < Ay$ are equivalent).

The algebra X is called **normable** if a measure with properties 1)–4) can be defined on it.

We shall say that the algebra X is **completely homogeneous** if there exists a group $\mathfrak{A} = \{A\}$ of automorphisms of this algebra such that:

I. For every $x \neq 0$,

$$\bigvee_{A \in \mathfrak{A}} Ax = 1.$$

II. If

$$\sum_{\alpha} x_{\alpha s} \to 0$$

as $s \to \infty$, then for arbitrary $A_{\alpha s} \in \mathfrak{A}$,

$$\bigvee_{\alpha} A_{\alpha s} x_{\alpha s} \to 0$$

as $s \to \infty$. Here the index α runs through an arbitrary set depending on s.

The first of these conditions means that the group $\mathfrak{A}$ is sufficiently large: for any $x, y \neq 0$ there is an $A \in \mathfrak{A}$ such that $Ax \wedge y \neq 0$.

The second condition, on the contrary, says that the automorphisms of the group $\mathfrak{A}$ are, in a certain sense, uniformly continuous.

A completely homogeneous algebra is always normable. Namely, the following holds.

Theorem 1. *If an algebra X has a group of automorphisms $\mathfrak{A}$ satisfying conditions I and II, then on X there exists a measure with properties 1)–4), invariant with respect to the automorphisms of the group $\mathfrak{A}$.*

An arbitrary normable algebra need not be completely homogeneous (an example is the discrete algebra of all subsets of a countable set). In the class of all algebras with measure, completely homogeneous algebras form a narrow but important subclass.

Let a be an arbitrary cardinal number, and let $\{I_\xi\}$ be a set of a copies of the interval $[0, 1]$. In the Cartesian product of all I_ξ, by the method described, for example, in (4), p. 156, one introduces a measure equal to the product of the Lebesgue measures on the I_ξ. The sets measurable with respect to this measure form a complete Boolean algebra (with equivalent sets identified). We shall denote this algebra by $P(a)$. Completely homogeneous algebras have precisely such a structure. More exactly, the following assertion is true:

Theorem 2. *For a complete Boolean algebra to be isomorphic to an algebra of the form $P(a)$, it is necessary and sufficient that it be infinite and completely homogeneous.*

It is known (see (5)) that as a one may take the minimal cardinality of a set dense in the given algebra (with respect to the order topology). The most important case is $a = \aleph_0$. For it the following holds.

Theorem 3. *For a complete Boolean algebra to be isomorphic to the algebra of Lebesgue-measurable sets of the interval (with the natural identification), it is necessary and sufficient that it be infinite, completely homogeneous, and separable.*

Here separability in the order topology is meant.

Finally, let us note that a finite algebra, as is easily seen, is always completely homogeneous.

The proof of Theorem 1 is carried out using the method of exhaustion. To construct the measure, the group $\mathfrak{A}$ is embedded in a broader group $\mathfrak{A}^*$ of automorphisms of the algebra such that, for any $x, y \in X$, one and only one of the following three relations holds:

a) for some $A^* \in \mathfrak{A}^*$, $A^*x < y$,

b) for some $A^* \in \mathfrak{A}^*$, $A^*x > y$,

c) for some $A^* \in \mathfrak{A}^*$, $A^*x = y$.

The possibility of such an embedding follows from conditions I, II, which the group $\mathfrak{A}^*$ will also satisfy. Next an "etalon" sequence $\{x_n\}$ is constructed such that

$$x_n - x_{n-1} = A_n^* x_n, \qquad A_n^* \in \mathfrak{A}^*, \qquad n = 1, 2, ...$$

After this the measure is defined by the formula

$$\mu(x) = \sum_k \frac{1}{2^{n_k}}, \quad \text{if } x = \sum_k{}' x'_{n_k}, \quad x'_{n_k} = \widetilde{A}'_{n_k} x_{n_k}, \quad \widetilde{A}'_{n_k} \in \mathfrak{A}^*.$$

It remains to verify the correctness of the definition and the presence of properties 1)–4).

The proof of Theorems 2 and 3 is based on Theorem 1, and also on the results of D. Maharam, who in (5) gave the necessary and sufficient condition for an algebra possessing a countably additive measure to be isomorphic to an algebra of the form $P(a)$* (see also (6)).

* It is necessary that all principal ideals of the algebra have one and the same topological weight. This, however, is not sufficient for the existence of a measure.

D. Maharam also established the possibility of representing any normed algebra as a union (direct sum) of algebras isomorphic to various $P(a)$. Hence, and from Theorem 2, it follows.

Theorem 4. *In order that a complete Boolean algebra be normable, it is necessary and sufficient that it be the union of at most a countable set of completely homogeneous components.*

A **component** or **principal ideal** of an algebra is, by definition, a set of the form $X_y = \{x; x \leqslant y\}$. This is also a Boolean algebra, in which y plays the role of the unit. One says that X is the union of the components

$$X_{y_n}, \quad \text{if } 1 = \sum_n y_n.$$

A Boolean algebra is called **regular** if it satisfies two conditions:

C. Every set of pairwise disjoint elements is at most countable.

D. If $x_n^m \geqslant x_{n+1}^m,\ n, m = 1, 2, \dots$, and $x_n^m \to 0$ as $n \to \infty,\ m = 1, 2, \dots$, then there exists a "diagonal" sequence $x_{n_m}^m$ for which

$$(o)\text{-}\lim_m x_{n_m}^m = 0.$$

The latter means that

$$\bigwedge_s \bigvee_{k>s} x_{n_k}^k = 0.$$

The concept of regularity was introduced by L. V. Kantorovich for linearly partially ordered spaces in 1936.*

A normable algebra is always regular (it is precisely this circumstance that underlies many facts in the theory of functions of a real variable, such as the theorems of D. F. Egorov, N. N. Luzin, etc.). Not a single example is known of a regular but non-normable algebra. However, an attempt to prove the normability of every regular algebra encounters, as shown in ([1]), major set-theoretic difficulties: this hypothesis is stronger than the known Suslin hypothesis.

Assuming in advance the regularity of the algebra X, one may weaken the requirements imposed on the group $\mathfrak{A}$. Namely, all sums mentioned in condition II may in this case be taken to be finite.

In the case of a regular algebra, the problem of constructing a measure is in general easier: it is enough to construct an essentially positive measure, and then, as A. G. Pinsker showed (([7]), p. 428), there also exists a countably additive measure**.

The conditions of complete homogeneity may be formulated by introducing into consideration a K-space "built over" the Boolean algebra under consideration (see ([7]), p. 128). Let X be an algebra satisfying condition C ("an algebra of countable type"); let X be an extended K-space for which X is a base. The following holds.

Theorem 5. *In order that an algebra X, satisfying condition C, be completely homogeneous, it is necessary and sufficient that it possess a group $\mathfrak{A}$ of automorphisms with properties I and III:*

III. *If*

$$\sum_{k=1}^{\infty} x_k = 1, \quad x_k \in X, \quad A_k \in \mathfrak{A}, \quad k = 1, 2, \dots, \quad \text{then} \quad \sum_{k=1}^{\infty} A_k x_k < +\infty$$

in the space X.

An analogous formulation, with ordinary series replaced by transfinite ones, is also possible for an algebra of uncountable type.

Let us note in conclusion that, at the cost of a certain complication of the formulations, in the above conditions of normability one may dispense with the requirement that the algebra be complete.

* Condition D is related to the condition of "weak countable distributivity" encountered in the literature. The latter, together with condition C, also gives regularity.
** This result was repeated by Kelley in ([2]).

The author expresses his gratitude to Prof. B. Z. Vulikh for his attention to the present work.

Leningrad State University
named after A. A. Zhdanov

Received
7 XII 1962

REFERENCES

[1] D. Maharam, Ann. of Math., 48, No. 1, 154 (1947).
[2] J. L. Kelley, Pacific J. Math., 9, No. 4, 1164 (1959).
[3] B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, Moscow, 1961.
[4] P. Halmos, *Measure Theory*, Moscow, 1953.
[5] D. Maharam, Proc. Nat. Acad. Sci., 28, 108 (1942).
[6] A. N. Kolmogorov, VI Zjazd Mat. Polskich, 1950, p. 2.
[7] L. V. Kantorovich, B. Z. Vulikh, A. G. Pinsker, *Functional Analysis in Partially Ordered Spaces*, Moscow–Leningrad, 1951.

Note: Figure translations are in progress. See original paper for figures.