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Abstract

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ON THE QUESTION OF THE EXISTENCE OF A COMMON BASIS IN NESTED SPACES OF ANALYTIC FUNCTIONS

(Presented by Academician A. N. Kolmogorov on 23 II 1962)

Let K' and K be two nondegenerate nested continua in the extended z-plane, each of which does not separate the plane; let $\mathfrak{A}(K')$ and $\mathfrak{A}(K)$ be the spaces of analytic functions regular, respectively, on the continua K' and K. The spaces $\mathfrak{A}(K')$ and $\mathfrak{A}(K)$ may, generally speaking, have a common basis. This will be so, for example, in any case when one of the continua is bounded by a regular analytic curve and the other is contained strictly inside it, as follows from the work of V. D. Erokhin ([1]). The latter assumption is essential for the method of the cited work. It is not known whether a common basis exists in the spaces under consideration when there is no strict inclusion of the continua. In the present note we give one particular result of a negative character on this question. Let, for definiteness, $K' \supset K$. We shall denote the complements of the corresponding continua by G' and G.

Definition. We shall call the continuum K a **metrically separable part** of the continuum K' if in its complement G there is a sequence of closed Jordan curves $\{\Gamma_n\}$, contracting to the boundary, such that, as $n \to \infty$, the harmonic measure of the set $\Gamma_n \cap K'$, relative to the domain interior to G and bounded by the curve Γ_n, tends to zero at some fixed point z_0.

In particular, as is not hard to see, every continuum contained in an arbitrary linear continuum of plane Lebesgue measure zero is its metrically separable part.

Theorem 1. *If the continuum K is a metrically separable part of the continuum K' and $K \neq K'$, then the spaces $\mathfrak{A}(K')$ and $\mathfrak{A}(K)$ do not have a common basis.*

Proof. Arguing by contradiction, we must assume that the conjugate spaces $\mathfrak{A}(G)$, $\mathfrak{A}(G')$, i.e. the spaces of functions regular inside the complements of the corresponding continua, also have a common basis. Without loss of generality one may assume that the domains G' and G are finite: after applying, if necessary, a conformal transformation to the domain G, we shall clearly preserve both the existence of a common basis in the corresponding transformed spaces of analytic functions and the property of metric separability of the complements of the transformed domains. We reduce the common basis of the spaces $\mathfrak{A}(G)$, $\mathfrak{A}(G')$ to canonical form as a basis of the space $\mathfrak{A}(G)$ ([2]). Then, if $f_j(z)$ are the basis functions ($j = 0, 1, \ldots$), and $F \subset G$ is an arbitrary closed set,

$$m_F = \lim_{j\to\infty} \left[\max_{z\in F} |f_j(z)|\right]^{1/j}, \qquad M_F = \lim_{j\to\infty} \left[\max_{z\in F} |f_j(z)|\right]^{1/j},$$

then, according to ([2]),

$$m_F \le M_F < 1,$$

while

$$\sup_{F\subset G} m_F = \sup_{F\subset G} M_F = 1.$$

Obviously, for an arbitrary closed set $F' \subset G'$ the inequality

$$m_{F'} \le M_{F'} < 1,$$

must hold, and, consequently,

$$\sup_{F'\subset G'} m_{F'} \le 1.$$

In the last relation equality does not occur, since in the contrary case it would not be difficult to see that the expansion in the basis of an arbitrary function $f(z) \in \mathfrak{A}(G')$ would have to converge uniformly inside the whole domain G, whereas $f(z)$ may have a singularity in this domain. Suppose that

$$\sup_{F'\subset G'} m_{F'} < \rho < 1.$$

We shall also bring this supposition to a contradiction. For this purpose choose a set $F \subset G$ such that

$$m_F > r > \rho.$$

If n is sufficiently large, then the set F is contained in any domain interior to G and bounded by the curve Γ_n. Map each such domain conformally onto the disk $|\zeta| < 1$ by means of the function $\zeta = \varphi_n(z)$ ($\varphi_n(z_0) = 0$, $\varphi_n'(z_0) > 0$), noting that the distance of the image of the set F from the boundary of the disk, for every n, will exceed some number $\delta > 0$, as is easily verified with the aid of Carathéodory' s theorem ([3]), p. 380). Let E_n be the closed set of points of the unit circle–the image of the set $\Gamma_n \cap K'$ under the corresponding conformal mapping. In view of the metric separability of the continuum K, for some n the inequality will be valid:

$$\operatorname{mes} E_n \leq \pi\delta^2 \left(1 - \frac{\ln r}{\ln \rho}\right),$$

where the measure is already understood in the sense of Lebesgue. Fixing n, construct a sequence of functions $\{u_j(\zeta)\}$, harmonic in the disk $|\zeta| < 1$ and continuous in the closed disk, taking on the boundary the values

$$u_j(\zeta) = \max\left\{\frac{1}{j} \ln \left|f_j(\varphi_n^{-1}(\zeta))\right|;\ \ln \rho\right\},$$

where $\varphi_n^{-1}(\zeta)$ is the inverse function with respect to the function $\zeta = \varphi_n(z)$. Cover the boundary set E_n by a system of open nonintersecting arcs E of total length not exceeding $2 \operatorname{mes} E_n$. The harmonic functions inside the disk will satisfy the inequality:

$$u_j(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} u_j(\tau) \frac{1 - |\zeta|^2}{|\tau - \zeta|^2}\, d\tau \leq \max_{\tau \in C(E)} u_j(\tau) \frac{1}{2\pi} \int_{C(E)} \frac{1 - |\zeta|^2}{|\tau - \zeta|^2}\, d\tau +$$

$$+ \max_{|\tau|=1} u_j(\tau) \frac{1}{2\pi} \int_E \frac{1 - |\zeta|^2}{|\tau - \zeta|^2}\, d\tau.$$

By virtue of the initial assumption, it is possible to choose a subsequence of the basis functions so that, for the corresponding values of j, each function $u_j(\zeta)$ does not exceed the number $\ln \rho$ on the set $C(E)$, while remaining

nonpositive on the whole circle. Fixing ξ so that the corresponding point z belongs to the set F, for the indicated values of j we shall have:

$$u_j(\xi) \leq \frac{\ln \rho}{2\pi} \int_{C(E)} \frac{1 - |\xi|^2}{|\tau - \xi|^2}\, d\tau = \ln \rho \left(1 - \frac{1}{2\pi} \int_E \frac{1 - |\xi|^2}{|\tau - \xi|^2}\, d\tau\right) \leq$$

$$\leq \ln \rho \left(1 - \frac{\operatorname{mes} E_n}{\pi \delta^2}\right) \leq \ln r,$$

and, consequently,

$$\max_{z \in F} u_j(\varphi_n(z)) \leq \ln r.$$

The same estimate is valid for the subharmonic functions $h_j(z) = \dfrac{1}{j} \ln |f_j(z)|$ (j belongs to the same sequence of values), for which the corresponding functions $u_j(\varphi_n(z))$ serve as harmonic majorants. The last conclusion contradicts the choice of the set F.

The theorem just proved can be given the following equivalent form.

Theorem $1'$**.** *If G' and G are two nested simply connected domains with metrically separated complements, then the spaces $\mathfrak{A}(G')$ and $\mathfrak{A}(G)$ do not have a common basis.*

This will be the case, for example, if G is the interior of a circle, while G' belongs to G and contains all points of G except the points of a radius, or the points of some closed circle whose boundary has internal tangency with the boundary of G.

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CITED LITERATURE

[1] V. D. Erokhin, DAN, **120**, No. 4 (1958).
[2] M. M. Dragilev, UMN, **15**, no. 2 (92) (1960).
[3] A. I. Markushevich, *Theory of Analytic Functions*, 1950.

Note: Figure translations are in progress. See original paper for figures.