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A. A. GALEEV, V. N. ORAEVSKII

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Abstract

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A. A. GALEEV, V. N. ORAEVSKII

ON THE INSTABILITY OF ALFVÉN WAVES

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It is known that in magnetohydrodynamics (and not only in an incompressible fluid, but also in a gas) Alfvén waves are exact solutions of the nonlinear equations. Therefore one might imagine that these waves exist for an indefinitely long time without a change of form and are stable in this sense. In fact, as will be shown here, they are unstable with respect to a certain type of perturbation, which contains not only Alfvén waves but also magnetosonic waves. We shall carry out the stability investigation by a method analogous to that used in work ([1]).

The stationary state is an Alfvén wave for which the magnetic field $\mathbf{H}\cos\mathbf{k}_0\mathbf{r}'$ (where $\mathbf{r}' = \mathbf{r} - \mathbf{u}t$, $\mathbf{u}$ is the wave velocity) and the hydrodynamic velocity $\mathbf{V}\cos\mathbf{k}_0\mathbf{r}'$ are perpendicular to the direction of propagation $\mathbf{k}_0$ and to the unperturbed magnetic field $\mathbf{H}_0$. In a system moving with the velocity of the wave, the linear equations for the perturbations have the form:

$$M\left[\frac{\partial}{\partial t} + \cos\mathbf{k}_0\mathbf{r}'\,(\mathbf{V}\nabla)\right]\mathbf{v} - MV\,(\mathbf{k}_0\mathbf{v})\sin\mathbf{k}_0\mathbf{r}' = \tag{1}$$

$$= \frac{1}{4\pi n_0}\Big\{[\operatorname{rot}\mathbf{h}, \mathbf{H}_0] + \frac{n}{n_0}\sin\mathbf{k}_0\mathbf{r}'\,[[\mathbf{k}_0\mathbf{H}]\mathbf{H}_0] + \cos\mathbf{k}_0\mathbf{r}'\,[\operatorname{rot}\mathbf{h}, \mathbf{H}] -$$

$$-\sin\mathbf{k}_0\mathbf{r}'\,[[\mathbf{k}_0\mathbf{H}]\mathbf{h}]\Big\} - \frac{1}{n_0}\nabla p, \qquad p = \gamma p_0\frac{n}{n_0},$$

$$\frac{\partial\mathbf{h}}{\partial t} = \operatorname{rot}[\mathbf{v}\mathbf{H}_0] + \operatorname{rot}[\mathbf{V}\cos\mathbf{k}_0\mathbf{r}'\,\mathbf{h}] + \operatorname{rot}[\mathbf{v}, \mathbf{H}\cos\mathbf{k}_0\mathbf{r}'],$$

$$\frac{\partial n}{\partial t} + n_0\operatorname{div}\mathbf{v} + \operatorname{div}(n\mathbf{V}\cos\mathbf{k}_0\mathbf{r}') = 0.$$

Here $\mathbf{v}, \mathbf{h}, n, p$ are the perturbations of velocity, magnetic field, density, and pressure, respectively.

In the coordinate system moving together with the wave, the coefficients in the magnetohydrodynamic equations (as is seen from (1)) do not depend on time. Consequently, the dependence of $\mathbf{v}, \mathbf{h}, n, p$ on time can be represented in the form $e^{-i\omega t}$. Thus the problem reduces to the solution of a system of equations which symbolically can be written as follows:

$$(\hat{\mathscr{H}}_0 + \hat{\mathscr{H}}_1)\varphi = \Omega\varphi, \tag{2}$$

where $\hat{\mathscr{H}}_0$ is a linear self-adjoint differential operator describing oscillations of a homogeneous plasma with eigenfunctions φ_Ω, whose spatial dependence is determined by factors $\exp(i\mathbf{kr}')$, and with real eigenvalues $\Omega^{(0)}$, satisfying the dispersion equation $\Omega^{(0)} = \Omega^{(0)}(\mathbf{k})$. $\hat{\mathscr{H}}_1$ is a linear differential operator with periodic coefficients, with $\hat{\mathscr{H}}_1 \to 0$ as $\mathbf{H}$ and $\mathbf{V} \to 0$. Therefore the stability of small-amplitude waves can be studied by the method of perturbation theory (the small parameter $\alpha \equiv V/u$).

To study stability it is necessary to find the correction $\omega^{(1)}$ to the eigenfrequency $\Omega^{(0)}$. As is known, $\omega^{(1)}$ in first-order perturbation theory is proportional to the matrix element $\langle\varphi_\Omega|\hat{\mathscr{H}}_1|\varphi_\Omega\rangle$. Taking into account the spatial dependence of $\hat{\mathscr{H}}_1$ (which, as is easy to see, is determined by the factors $\exp(\pm i\mathbf{k}_0\mathbf{r}'))$, one may say that $\omega^{(1)}$ is nonzero only in the case when one $\Omega^{(0)}$ corresponds to at least two wave vectors, different in magnitude and related to one another by

Fig. 1

Fig. 1

$$\mathbf{k}_1 = \mathbf{k}_0 + \mathbf{k}_2. \tag{3}$$

In the laboratory coordinate system relation (3) is unchanged, while the frequencies $\omega_1^{(0)}$ and $\omega_2^{(0)}$, corresponding to the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, are related in this coordinate system by

$$\omega_1^{(0)} = \omega_0 + \omega_2^{(0)}, \tag{4}$$

where ω_0 is the frequency of the oscillations of the "background." Thus, instability of Alfvén waves can arise (in first-order perturbation theory) only for perturbations having the form of a sum of two waves for which conditions (3), (4) are satisfied.

Using (1), (3), and (4), one can obtain the following system of algebraic equations for the wave amplitudes:

$$\begin{gathered}2\left(-\omega_{1,2}\mathbf{v}_{1,2} - \frac{1}{4\pi n_0 M}[[\mathbf{k}_{1,2}\mathbf{h}_{1,2}]\mathbf{H}_0] + v_s^2\frac{n_{1,2}}{n_0}\mathbf{k}_{1,2}\right)\\ = -(\mathbf{V}\mathbf{k}_{2,1})\mathbf{v}_{2,1} \mp (\mathbf{v}_{2,1}\mathbf{k}_0)\mathbf{V} + \frac{1}{4\pi n_0 M}\Big\{[[\mathbf{k}_{2,1}\mathbf{h}_{2,1}]\mathbf{H}]\\ \pm [[\mathbf{k}_0\mathbf{H}]\mathbf{h}_{2,1}] \mp [[\mathbf{k}_0\mathbf{H}]\mathbf{H}_0]\frac{n_{2,1}}{n_0}\Big\},\end{gathered} \tag{5}$$

$$-\omega_{1,2}\mathbf{h}_{1,2} - [\mathbf{k}_{1,2}[\mathbf{v}_{1,2}\mathbf{H}_0]] = \frac{1}{2}\,[\mathbf{k}_{1,2}\{[\mathbf{V}\mathbf{h}_{2,1}] + [\mathbf{v}_{2,1}\mathbf{H}]\}],$$

$$-\omega_{1,2}n_{1,2} + n_0(\mathbf{k}_{1,2}\mathbf{v}_{1,2}) + \frac{1}{2}n_{2,1}(\mathbf{k}_{1,2}\mathbf{V}) = 0,$$

$$v_s^2 = \gamma\frac{p_0}{Mn_0},$$

where $\mathbf{h}_{1,2}$, $\mathbf{v}_{1,2}$, $n_{1,2}$ are the amplitudes of waves with frequencies $\omega_1 = \omega_1^{(0)} + \omega^{(1)}$, $\omega_2 = \omega_2^{(0)} + \omega^{(1)}$ (where $\omega^{(1)}$ is the correction to the frequencies, still related by relation (4)) and with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. Using the system of equations (5), one can investigate the stability of an Alfvén wave with respect to various kinds of perturbations.

Let the perturbation be a combination of an Alfvén wave and a magnetosonic wave, denoted below respectively by the indices $1, 2$. Then from the solvability condition of system (5) we find the correction $\omega^{(1)}$

$$\omega^{(1)2} = \frac{k_{2y}^2 V^2}{16\left[1 + \left(\dfrac{k_{2x}k_{2y}v_s^2}{\omega_2^2 - k_{2x}^2 v_s^2}\right)^2\right]}\left\{\frac{\omega_2^2\cos^2\delta}{\omega_2^2 - k_{2x}^2 v_s^2} - 4\sin\gamma\sin(\gamma+\delta)\right\}\frac{\omega_1}{\omega_2}, \quad (6)$$

where δ is the angle between the planes $(\mathbf{k}_0, \mathbf{H}_0)$ and $(\mathbf{k}_1, \mathbf{H}_0)$, and γ is the angle between $(\mathbf{k}_1, \mathbf{H}_0)$ and $(\mathbf{k}_2, \mathbf{H}_0)$; the x-axis is chosen along $\mathbf{H}_0$, and the y-axis perpendicular to $\mathbf{H}_0$ in the plane $(\mathbf{k}_2, \mathbf{H}_0)$ (see Fig. 1); $\mathbf{k}_1$, $\mathbf{k}_2$, ω_1, ω_2 satisfy conditions (3) and (4).

Using (3), (4), (6), it can be shown that the initial Alfvén wave is unstable for any angle of propagation with respect to the magnetic field. For the case of a strong magnetic field

$$\frac{H_0^2}{8\pi} \gg p_0. \quad (7)$$

The increment $\nu = -i\omega^{(1)}$ of the growth of small perturbations in the form of Alfvén and slow magnetosonic waves is, in order of magnitude, equal to

$$\nu \simeq \frac{V}{(8u_s)^{1/2}}\,\omega_0. \quad (8)$$

We note that the increment is proportional to the wave amplitude.

If, instead of an Alfvén wave, a fast magnetosonic wave enters into the perturbation, then it can be shown that the increment of growth of the perturbations is of the same order as ν.

Perturbations from two Alfvén waves, as is seen from (3), (4), propagate in the same direction relative to $\mathbf{H}_0$; from (5) it is seen that these waves do not interact with one another. The growth increments of other types of perturbations, however, are much smaller than ν.

Thus, from the consideration carried out above it is seen that Alfvén waves, over times of order $1/\nu$, turn into disordered oscillations of the medium. This means, in particular, that the assumption of a sufficiently long existence of Alfvén waves, which underlies some astrophysical and geophysical hypotheses, is incorrect.

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Novosibirsk State University

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Note: Figure translations are in progress. See original paper for figures.