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Abstract

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MATHEMATICS

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ON CONDITIONS FOR SELF-ADJOINTNESS OF DIFFERENTIAL OPERATORS OF HIGHER ORDER

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The present note is devoted to the proof of the following theorem.

Theorem. Let L be the minimal operator defined in the space $\mathcal{L}_2(-\infty, \infty)$ by the differential expression *

$$ly = (-1)^n \frac{d^{2n}y}{dx^{2n}} + q(x)y, \tag{1}$$

where $q(x)$ is a continuous real-valued function on $(-\infty, \infty)$. Suppose that there exists a sequence of intervals Δ_k of length δ_k and a sequence of numbers $\gamma_k > 1$ $(k = \pm 1, \pm 2, ...)$ such that: a) as $k \to +\infty$ the interval Δ_k tends to $+\infty$, and as $k \to -\infty$ to $-\infty$; b) if $x \in \Delta_k$, then $q(x) > -C\gamma_k$ (C does not depend on k); c) the series

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_k + \delta_k^{-2n}}$$

and

$$\sum_{k=-\infty}^{1} \frac{1}{\gamma_k + \delta_k^{-2n}}$$

diverge. Then the operator is self-adjoint (i.e. has deficiency index $(0, 0)$).

Let us note two special cases of the theorem.

1. If $q(x) > -CQ(|x|)$, where $Q(t)$ is a nondecreasing function, and

$$\int_0^{\infty} |Q(t)|^{-1+1/2n} \, dt = \infty,$$

then L is self-adjoint.

As we shall show below, this assertion is easily derived from the theorem. For $n = 1$ this fact was proved by Sears ([1]).

2. If $q(x) > -C$ on a sequence of intervals Δ_k of length δ_k, where Δ_k tends to $+\infty$ as $k \to +\infty$ and to $-\infty$ as $k \to -\infty$, and

$$\sum_{k=1}^{\infty} \delta_k^{2n} = \sum_{k=-\infty}^{1} \delta_k^{2n} = \infty,$$

then L is self-adjoint.

We remark that for $n = 1$ the class of self-adjoint operators L described in the theorem is close to the class found by I. Brinck in ([2]); however, these classes do not coincide.

We proceed to the proof. In what follows, by A_i, B_i, C_i we shall denote absolute constants. Denote by L^* the operator adjoint to L, and by $D(L^*)$ its domain.

Lemma 1. Let $\Delta_k = (a_k, b_k)$ $(k = \pm 1, \pm 2, ...)$ be a sequence of nonintersecting intervals such that Δ_k tends to $+\infty$ as $k \to +\infty$ and to $-\infty$ as $k \to -\infty$; further, let $\delta_k = b_k - a_k$. If for every function $y(x) \in D(L^*)$

$$\sum_{i=0}^{n} \delta_k^{-2n+2i} \int_{\Delta_k} |y^{(i)}(x)|^2 \, dx \to 0 \quad \text{as } |k| \to \infty, \tag{2}$$

then the operator L is self-adjoint.

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* The minimal operator defined by expression (1) is called the closure of the operator L_0, given by the equality $L_0 y = ly$ on finite smooth functions.

Proof. For each $k = \pm 1, \pm 2, ...$ construct a smooth function $\varphi_k(x)$ such that $0 \le \varphi_k(x) \le 1$, $\varphi_k(x) = 0$ outside the interval (a_{-k}, b_k); $\varphi_k(x) = 1$ on the interval (b_{-k}, a_k), and $|\varphi_k^{(i)}(x)| < C\delta_k^{-i}$ for all x and $0 \le i \le 2n$. Put $\psi_k(x) \equiv 1 - \varphi_k(x)$.

Let $u \in D(L^*)$ and $v \in D(L^*)$. We shall show that $(L^*u, v) = (u, L^*v)$,

$$(L^*u, v) - (u, L^*v) = (l(u\varphi_k) + l(u\psi_k), v) - (u, lv) = (l(u\varphi_k), v) - (u, lv) + (l(u\psi_k), v).$$

Since $u\varphi_k$ is a finite function, it follows that

$$(l(u\varphi_k), v) - (u, lv) = (u\varphi_k, lv) - (u, lv) = -(u\psi_k, lv) \to 0$$

as $k \to \infty$. It remains to show that $(l(u\psi_k), v) \to 0$,

$$(l(u\psi_k), v) = (\psi_k \cdot lu, v) + \int_{\Delta_k} (-1)^n \sum_{i=1}^{2n} C_{2n}^i u^{(2n-i)} \psi_k^{(i)} v \, dx.$$

Obviously, $(\psi_k \cdot lu, v) \to 0$ as $|k| \to \infty$.

We transform the second term by integration by parts and estimate it by the inequality $2|ab| \leq a^2 + b^2$:

$$\left| \int_{\Delta_k} (-1)^n \sum_{i=1}^{2n} C_{2n}^i u^{(2n-i)} \psi_k^{(i)} v \, dx \right| =$$

$$= \left| \sum_{i=1}^{n} \int_{\Delta_k} A_i \psi_k^{(2n-2i)} u^{(i)} v^{(i)} + B_i \psi_k^{(2n-2i-1)} u^{(i)} v^{(i)} \, dx \right| \leq$$

$$\leq C \sum_{i=0}^{n} \delta_k^{-2n+2i} \int_{\Delta_k} (|u^{(i)}|^2 + |v^{(i)}|^2) \, dx \to 0.$$

Thus, $(L^* u, v) = (u, L^* v)$ for all $u, v \in D(L^*)$; but then $L^* = L$. The lemma is proved.

For what follows, fix an interval $\Delta = (a, b)$ of length $h = b - a$. Take the function

$$\sigma_k(x) = (x-a)^{2n-2k} (b-x)^{2n-2k}$$

and put, for an arbitrary function $y(x)$ with $2n$ continuous derivatives,

$$B_i(y) = h^{2n-2i} \int_\Delta \sigma_{n-i}(x) |y^{(i)}(x)|^2 \, dx.$$

Lemma 2. If $p > 1$, then

$$B_i(y) > p B_{i-1}(y) - C_0 p^2 B_{i-2}(y). \tag{3}$$

Proof. The identity

$$\int_\Delta \left[(x-a)^m (b-x)^m z'' + \frac{1}{2} p h^2 (x-a)^{m-2} (b-x)^{m-2} z \right]^2 dx =$$

$$= \int_\Delta \Big\{ (x-a)^{2m} (b-x)^{2m} |z''|^2 - p h^2 (x-a)^{2m-2} (b-x)^{2m-2} {z'}^2 +$$

$$+ \frac{p^2 h^4}{4} (x-a)^{2m-4} (b-x)^{2m-4} \left[h^2 + \frac{\Phi_2(x)}{p} \right] z^2 \Big\} \, dx,$$

where $\Phi_2(x)$ is a polynomial of second degree, and $|\Phi_2(x)| < ch^2$. Hence

$$\int_\Delta (x-a)^{2m} (b-x)^{2m} |z''|^2 \, dx \geq p h^2 \int_\Delta (x-a)^{2m-2} (b-x)^{2m-2} |z'|^2 \, dx -$$

$$- c p^2 h^4 \int_\Delta (x-a)^{2m-4} (b-x)^{2m-4} |z|^2 \, dx.$$

Putting $z = y^{(m-2)}$, we obtain from this inequality (3).

Lemma 3. Suppose that on the interval $\Delta = (a, b)$ of length $h = b - a$ the inequality $q(x) > -Cy$ holds. Denote the interval $(a + h/3, b - h/3)$ by-offsetof through Δ_1. Then for any $y(x)$

$$\sum_{i=0}^{n} \int_{\Delta_1} h^{-2n+2i} \left|y^{(i)}(x)\right|^2 \, dx \leq C_0 \int_{\Delta} \left[\gamma |ly|^2 + \left(\gamma + \frac{1}{\gamma} + h^{-2n}\right) |y|^2\right] \, dx. \quad (4)$$

Proof. Let $\tau(x) = \gamma^{-1} h^{-4n} \sigma_0(x)$, where $\sigma_0(x)$ is the function introduced above. From the identity

$$\int_{\Delta} |ly - \tau y|^2 \, dx = \int_{\Delta} |ly|^2 \, dx + \int_{\Delta} \tau^2 y^2 \, dx - 2 \int_{\Delta} \left[(\tau y)^{(n)} y^{(n)} + q \tau y^2\right] \, dx$$

we find

$$\int_{\Delta} (\tau y)^{(n)} y^{(n)} \, dx \leq \int_{\Delta} \left[|ly|^2 + \tau^2 y^2 - q \tau y^2\right] \, dx. \quad (5)$$

We transform the left-hand side of the inequality, integrating by parts:

$$\int_{\Delta} (\tau y)^{(n)} y^{(n)} \, dx = \frac{1}{\gamma h^{4n}} \int_{\Delta} \sigma_0(x) \left|y^{(n)}\right|^2 \, dx + \int_{\Delta} \sum_{i=0}^{n-1} C_i \sigma_0^{(2n-2i)} \left|y^{(i)}\right|^2 \, dx.$$

It is easy to show that $\sigma_0^{(2n-2i)} = \sigma_{n-i} P_i(x)$, where $P_i(x)$ is a polynomial of degree $2n - 2i$, and $|P_i(x)| < B_0 h^{2n-2i}$. Therefore

$$\int_{\Delta} (\tau y)^{(n)} y^{(n)} \, dx > \frac{1}{\gamma h^{4n}} \int_{\Delta} \left[\sigma_0 \left|y^{(n)}\right|^2 - C_2 \sum_{i=0}^{n-1} h^{2n-2i} \sigma_{n-i} \left|y^{(i)}\right|^2\right] \, dx =$$

$$= \frac{1}{\gamma h^{4n}} \left[B_n(y) - C_2 \sum_{i=0}^{n-1} B_i(y)\right]. \quad (6)$$

Put $i = n$, $p = C_2 + 1$ in inequality (3), and substitute the estimate thereby obtained for $B_n(y)$ into (6). We obtain

$$\int_\Delta (\tau y)^{(n)} y^{(n)}\, dx > \frac{1}{\gamma h^{4n}} \left[B_{n-1}(y) - C_3 \sum_{i=0}^{n-2} B_i(y) \right].$$

In exactly the same way, using inequality (3) (for $i = n-1,\ p = C_3 + 1$), we exclude from the right-hand side of the last inequality the term $B_{n-1}(y)$, and so on. As a result we obtain a chain of inequalities of the form

$$\int_\Delta (\tau y)^{(n)} y^{(n)}\, dx > \frac{1}{\gamma h^{4n}} \left[B_k(y) - A_0 \sum_{i=0}^{k-1} B_i(y) \right] \qquad (k = 1, \dots, n-1).$$

From these inequalities we obtain successively

$$\frac{1}{\gamma h^{4n}} \left[B_i(y) - A_1 B_0(y) \right] \le \int_\Delta (\tau y)^{(n)} y^{(n)}\, dx \qquad (i = 1, \dots, n). \tag{7}$$

But

$$B_i(y) = h^{2n-2i} \int_\Delta \sigma_{n-i} \left| y^{(i)} \right|^2 dx \ge h^{2n-2i} \int_{\Delta_1} \sigma_{n-i} \left| y^{(i)} \right|^2 dx \ge$$

$$\ge c h^{2n+2i} \int_{\Delta_1} \left| y^{(i)} \right|^2 dx \quad \text{for } i = 1, 2, \dots, n; \qquad B_0(y) = h^{2n} \int_\Delta |y|^2\, dx.$$

Therefore, from (7) we find

$$\frac{1}{\gamma} c h^{-2n+2i} \int_{\Delta_1} \left| y^{(i)} \right|^2 dx - \frac{1}{\gamma h^{2n}} \int_\Delta |y|^2\, dx \le \int_\Delta (\tau y)^{(n)} y^{(n)}\, dx \qquad (i = 1, \dots, n).$$

Adding these inequalities, we obtain

$$\frac{C_1}{\gamma} \sum_{i=0}^{n} h^{-2n+2i} \int_{\Delta_1} |y^{(i)}|^2\, dx - \frac{C_2}{\gamma} h^{-2n} \int_\Delta |y|^2\, dx \le \int_\Delta (\tau y)^{(n)} y^{(n)}\, dx. \tag{8}$$

It is obvious that the right-hand side of inequality (5) does not exceed the sum

$$C_3 \int_\Delta \left[(ly)^2 + \frac{1}{\gamma^2} y^2 + y^2 \right] dx.$$

Hence, from inequality (8) we obtain inequalities (4). The lemma is proved.

Proof of the theorem. Let Δ_k $(k = \pm 1, \pm 2, ...)$ be the intervals specified in the condition of the theorem. From Lemmas 1 and 3 it follows that the operator L is self-adjoint if

$$\lim_{k\to\infty} \int_{\Delta_k} \left[(ly)^2 \gamma_k + \left(\frac{1}{\gamma_k} + \gamma_k + \delta_k^{-2n} \right) y^2 \right] dx =$$

$$= \lim_{k\to -\infty} \int_{\Delta_k} \left[(ly)^2 \gamma_k + \left(\frac{1}{\gamma_k} + \gamma_k + \delta_k^{-2n} \right) y^2 \right] dx = 0.$$

Consider, for example, the case $k \to +\infty$. We have

$$\int \left[(ly)^2 \gamma_k + \left(\frac{1}{\gamma_k} + \gamma_k + \delta_k^{-2n} \right) y^2 \right] dx \le c(\gamma_k + \delta_k^{-2n}) \int_{\Delta_k} [y^2 + (ly)^2] \, dx$$

(we have used the fact that $\gamma_k > 1$). If we assume that

$$(\gamma_k + \delta_k^{-2n}) \int_{\Delta_k} [y^2 + (ly)^2] \, dx > \varepsilon_0$$

for all $k > k_0$, then we obtain

$$\frac{\varepsilon_0}{\gamma_k + h_k^{-2n}} < \int_{\Delta_k} [y^2 + (ly)^2] \, dx.$$

But then, since $y \in L_2$, $ly \in L_2$,

$$\sum_{k=1}^{\infty} \frac{\varepsilon_0}{\gamma_k + h_k^{-2n}} < \sum_{k=1}^{\infty} \int_{\Delta_k} [y^2 + (ly)^2] \, dx < \infty,$$

which contradicts the condition of the theorem. The theorem is proved.

It is easy to show that condition c) of the theorem may be replaced (with the other conditions retained) by the following:

c') $\quad \delta_k \gamma_k^{1/2n} > \varepsilon_0 \quad (k = \pm 1, \pm 2, ...); \qquad \sum_{k=1}^{\infty} \delta_k \gamma_k^{-1+1/2n} = \sum_{k=-\infty}^{1} \delta_k \gamma_k^{-1+1/2n} = \infty.$

For the proof one should divide each interval Δ_k into

$$\left[\frac{1}{2\varepsilon_0}\delta_k\gamma_k^{1/2n}\right]$$

equal parts and apply the theorem to the intervals Δ_k' thus obtained; in doing so, condition c) for Δ_k' passes into condition c) for Δ_k.

From this remark it is easy to obtain the generalization of Sears' s theorem to higher-order operators mentioned above; for this it suffices to put $\delta_k = (k-1, k)$, $\gamma_k = Q(x_k)$.

We note in conclusion that the arguments presented in this note are easily transferred to partial differential operators of the form

$$L = \partial^{2m}/\partial x^{2m} + \partial^{2m}/\partial y^{2m} + q(x, y).$$

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CITED LITERATURE

[1] D. B. Sears, Canad. J. Math., 2, 314 (1950).

[2] I. Brinck, Math. Scand., 7, No. 1, 219 (1959).

Note: Figure translations are in progress. See original paper for figures.