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Abstract

Full Text

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ON THE EXTENSION OF THE GOLUZIN-SCHIFFER VARIATIONAL METHOD TO MULTIPLY CONNECTED DOMAINS

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1. Denote by D an n-connected domain of the z-plane, obtained by removing from the annulus $1 < |z| < R$ the interiors of $n-2$ nondegenerate circles L_k $(k = 2, \dots, n-1;\ L_1 : |z| < 1,\ L_n : |z| < R)$.

Let F be the class of functions $w = f(z)$ possessing the following properties: 1) $f(z)$ is regular and univalent in D; 2) $f(z_0) = 1$, where z_0 is a fixed point of the interval $(1, R)$; 3) $f(z) \neq 0,\ z \in D$; 4) if the curve l_1 is contained inside the curve l_2, l_1 and $l_2 \subset D$, then the curve fl_1 is contained inside the curve fl_2.

Let, further, $\Phi[f]$ be a real functional, defined for all functions analytic in the domain D, and such that for any function $g(z)$ analytic in D the relation (1) holds

$$\Phi[f + \varepsilon g] = \Phi[f] + \mathrm{Re}\{\varepsilon \Psi[f, g]\} = O(\varepsilon^2), \tag{1}$$

where $\Psi[f, g]$ is a complex functional, linear with respect to $g(z)$. Suppose that $\Phi[f]$ has a finite upper bound in the class F.

We pose the problem: *to determine the functions of the class F for which $\Phi[f]$ assumes its greatest value.*

2. The existence of extremal functions follows from the normality of the family F in the domain D.

We shall agree to denote by $D(L)$ such a simply connected domain which contains D and has as its exact boundary the closed set L contained in the boundary of the domain D. The following theorem is known (^{2}):

Let in a domain D of the z-plane there be given a univalent and single-valued conformal mapping $w = f(z)$ of the domain D onto the w-plane. For any finite decomposition of the boundary $\Gamma = L_1 \cup L_2 \cup \dots \cup L_n$ of the domain D into nonempty pairwise disjoint closed sets there is a decomposition of f into a superposition

$$f(z) = f_n f_{n-1} \cdots f_1(z), \qquad z \in D, \tag{2}$$

where the mappings $f_1, f_2, \ldots, f_n$ are one-to-one and conformal respectively in the domains $D(L_1)$, $f_1 D(f_1 L_2), \ldots,$ $f_{n-1} \cdots f_1 D(f_{n-1} \cdots f_1 L_n)$; such a decomposition is unique up to intermediate linear-fractional transformations.

3. We shall represent the extremal function $f(z)$ of our problem by formula (2). We choose the component $t_1 = f_1(z)$ so that the conditions $f_1(\infty) = \infty$, $f_1(z) \neq 0$, $f_1(z_0) = t_1^0$ are satisfied, where t_1^0 is determined from the relation $f_n f_{n-1} \cdots f_2(t_1^0) = 1$.

Using the variational formula of G. M. Goluzin ((^{3}), formula (2), p. 125), we obtain a variation $\tilde{f}_1(z)$ for $f_1(z)$; then, after a suitable normalization, the function $f_n f_{n-1} \cdots f_2 \tilde{f}_1(z)$ will be a varied func-

functions in the class F. The variational formulas have the form

$$\tilde{f} = f + h\frac{f'}{f_1'}\frac{f_1(f_1^0 - f_1)}{(f_1 - t_1^*)(f_1^0 - t_1^*)} + O(\lambda^2), \tag{3}$$

where t_1^* is an exterior point with respect to the domain $f_1 D(f_1 L_1)$;

$$\begin{split}\tilde{f} = f + \frac{f'}{f_1'}\Bigg[-h\,\frac{f_1^{*2}(f_1 - f_1^0)}{(f_1^* - f_1)(f_1^* - f_1^0)} + h\,\frac{f_1^{*3}}{z^{*2} f_1^{*\prime 2}}\left(\frac{z^2 f_1'}{z^2 - z} - \frac{f_1}{f_1^0}\frac{z_0^2 f_1^{0\prime}}{z^* - z_0}\right)\\ - \left(\frac{f_1^{*3}}{z^{*2} f_1^{*\prime 2}}\right)\left(\frac{z f_1'}{z^* z - 1} - \frac{f_1}{f_1^0}\frac{z_0 f_1^{0\prime}}{z^* z_0 - 1}\right)\Bigg] + O(\lambda^2),\end{split} \tag{4}$$

where $\tilde{f} = \tilde{f}(z)$, $f = f(z)$, $f' = f'(z)$, $f_1 = f_1(z)$, $f_1' = f_1'(z)$, $f_1^* = f_1(z^*)$, $f_1^{*\prime} = f_1'(z^*)$, $f_1^0 = f_1(z_0)$, $f_1^{0\prime} = f_1'(z_0)$, $z^* \in D$, $h = \lambda e^{i\alpha}$.

We shall have

$$\Phi[\tilde{f}] \leqslant \Phi[f].$$

Hence, taking into account the arbitrariness of $\arg h$, in the usual way we obtain for f_1^* the differential equation

$$\begin{split}\Psi\left[f, \frac{f'}{f_1'}\frac{f_1(f_1 - f_1^0)}{f_1^* - f_1}\right]\frac{z^* f_1^{*\prime 2}}{f_1^*(f_1^* - f_1^0)} = -\Psi\left[f, \frac{z_0^2 f_1^{0\prime} f_1 f'}{f_1' f_1^0 (z^* - z_0)}\right] + \\ +\Psi\left[f, \frac{z^2 f'}{z^* - z}\right] + \Psi\left[f, \frac{z_0 f_1^{0\prime} f_1 f'}{f_1' f_1^0 (z^* z_0 - 1)}\right] - \Psi\left[f, \frac{z f'}{z^* z - 1}\right].\end{split} \tag{5}$$

Here f_1^* denotes the component f_1 of the extremal function in the superposition (2).

Consider the auxiliary function $\tilde{f}_\theta(z) \in F$:

$$\tilde{f}_\theta(z) = f_n f_{n-1} \dots f_2 \left(f_1(z_0) \frac{f_1(e^{i\theta} z)}{f_1(e^{i\theta} z_0)} \right) = f(z) + i\theta\Omega_1 + O(\theta^2),$$

where

$$\Omega_1 = \frac{f}{f_1'} \frac{z f_1' f_1^0 - z_0 f_1^{0\prime} f_1}{f_1^0},$$

θ is arbitrary and real. From the condition $\Phi[\tilde{f}_\theta] \leqslant \Phi[f]$ we find that $\operatorname{Im}\Omega_1 = 0$. Hence it is easy to see that the right-hand side of equation (5) is real for $|z^*| = 1$. Suppose further that

$$\Psi\left[f, \frac{f'}{f_k' f_{k-1}' \dots f_1'} \frac{f_k(f_k - f_k^0)}{f_k^* - f_k}\right]$$

is a meromorphic function of f_k^* for all $k = 1, 2, \dots, n$.

By the principle of symmetry we conclude that the left-hand side of (5) ($k = 1$) can be analytically continued into $|z| \leqslant 1$. This shows that $f_1 L_1$ consists of analytic arcs.

Using formula (3), we shall prove that the domain $f_1 D(f_1 L_1)$ has no exterior points and, consequently, is the whole f_1-plane with cuts along analytic arcs.

4. Let the function $\varphi_1(\zeta)$, $\varphi_1(\infty) = \infty$, map the domain $|\zeta| > 1$ onto the domain $f_1 D(f_1 L_2)$. The function $\varphi(\zeta)$ is analytic on $|\zeta| = 1$, since it maps the circle $|\zeta| = 1$ onto the analytic arc $f_1 L_2$ ((4), p. 164). We choose the component $t_2 = f_2(f_1)$ so that the conditions $f_2(\infty) = \infty$, $f_2(f_1^0) \neq 0$, $f_2(f_1^0 = t_2^0)$,

where t_2^0 is determined from the relation $f_n f_{n-1} \cdots f_3(t_2^0) = 1$. We vary the function $f_2(\varphi(\zeta))$, $|\zeta| > 1$, $f_1 = \varphi_1(\zeta)$; then, after a suitable normalization, the function $f_n f_{n-1} \cdots f_3 \tilde{f}_2(f_1)$ will be a varied function in the class F. Carrying out the computations, as above, we obtain a differential equation for the component f_2^* of the extremal function f:

$$\Psi\left[f, \frac{f'}{f_2' f_1'} \frac{f_2(f_2 - f_2^0)}{f_2^* - f_2}\right] \frac{\zeta^{*2} \varphi_1'^{\,2}(\zeta^*) f_2^{*\prime\,2}}{f_2^*(f_2^* - f_2^0)} =$$

$$= \Psi\left[f, \frac{\zeta^2 \varphi_1'(\zeta) f'}{f_1'(\zeta^* - \zeta)}\right] - \Psi\left[f, \frac{\zeta_0^2 f_2^0 \varphi_1'(\zeta_0) f_2 f'}{f_2' f_1' f_2^0 (\zeta^* - \zeta_0)}\right] +$$

$$+ \overline{\Psi\left[f, \frac{\zeta_0 \varphi_1'(\zeta_0) f_2^{0\prime} f_2 f'}{f_2' f_1' f_2^0 (\zeta^* \zeta_0 - 1)}\right]} \Psi - \overline{\Psi\left[f, \frac{\zeta \varphi_1'(\zeta) f'}{f_1'(\zeta^* \zeta - 1)}\right]}, \tag{6}$$

where we have put

$$f_2 = f_2(f_1), \quad f_2^0 = f_2(f_1^0), \quad f_2^* = f_2(f_1^*), \quad f_2' = f_2'(f_1), \quad f_2^0{}' = f_2'(f_1^0),$$
$$f_2^*{}' = f_2'(f_1^*), \quad \zeta = \varphi_1^{-1}(f_1), \quad \zeta_0 = \varphi_1^{-1}(f_1^0), \quad \zeta^* = \varphi_1^{-1}(f_1^*).$$

We define the auxiliary variation by the formula

$$\tilde{f}_\theta(z) = f_n f_{n-1} \cdots f_3 \left[f_2(\varphi_1(\zeta_0)) \frac{f_2(\varphi_1(e^{i\theta}\zeta))}{f_2(\varphi_1(e^{i\theta}\zeta_0))} \right] = f(z) + i\theta\Omega_2 + O(\theta^2),$$

where

$$\Omega_2 = \frac{f'}{f_2' f_1'} \frac{f_2^0 f_2 \zeta \varphi_1'(\zeta) - f_2^0{}' f_2 \zeta_0 \varphi_1'(\zeta_0)}{f_2^0},$$

θ is real and arbitrary.

From the condition $\Phi[\tilde{f}_\theta] \le \Phi[f]$ we find that $\operatorname{Im}\Omega_2 = 0$. Hence it is easy to see that the right-hand side of equation (6) is real for $|\zeta^*| = 1$. By the symmetry principle we conclude that the left-hand side of (6) can be analytically continued to $|\zeta| \le 1$; consequently, $f_2 f_1 L_2$ consists of analytic arcs. The domain $f_2 f_1 D\,(f_2 f_1 L_2)$ has no exterior points and is the whole f_2-plane with slits along analytic arcs.

5. Let the function $\varphi_k(\zeta)$, $\varphi_k(\infty) = \infty$, $k = 1, \dots, n-1$, map the domain $|\zeta| > 1$ onto the domain $f_k f_{k-1} \cdots f_1 D\,(f_k \cdots f_1 L_{k+1})$. The functions $\varphi_k(\zeta)$ $(k = 1, \dots, n-1)$ are analytic on $|\zeta| = 1$, since they map the circle $|\zeta| = 1$ onto the analytic arcs $f_k \cdots f_1 L_{k+1}$. We choose the component $t_{k+1} = f_{k+1}(f_k)$ so that the conditions $f_{k+1}(\infty) = \infty$ for $k = 1, \dots, n-2$ and $f_n(\infty) = 0$; $f_{k+1}(f_k) \ne 0$ $(k = 1, \dots, n-2)$; $f_{k+1}(f_k^0) = t_{k+1}^0$, where t_{k+1}^0 is determined from the relation $f_n \cdots f_{k+2}(t_{k+1}^0) = 1$, are satisfied. For the component f_{k+1}^* of the extremal function we obtain a differential equation analogous to (6), from which we conclude that $f_{k+1} \cdots f_1 L_{k+1}$ $(k = 1, \dots, n-1)$ consists of analytic arcs. The extremal function f maps the domain D onto the whole w-plane with n slits along piecewise-analytic arcs.

6. As an example, we consider the well-known problem of maximizing $|f'(z_0)|$ in the class F for $n = 2$. Without loss of generality one may assume $f'(z_0) > 0$, and then the problem reduces to determining the maximum of $\operatorname{Re} f'(z_0)$. In this case, for the extremal function one obtains the following explicit expression:

$$w = f(z) = \frac{\operatorname{cn} u_0 + \operatorname{dn} u_0}{\operatorname{cn} u_0 - \operatorname{dn} u_0} \frac{\operatorname{cn} u - \operatorname{dn} u}{\operatorname{cn} u + \operatorname{dn} u}, \tag{7}$$

where

$$u = \frac{2K}{\ln R} \ln z, \quad u_0 = \frac{2K}{\ln R} \ln z_0, \quad K = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \frac{2\pi}{\ln R} = \frac{K'}{K},$$

$$K' = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-{k'}^2x^2)}}, \quad k^2 + {k'}^2 = 1,$$

$\operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u$ are elliptic func-

Jacobi functions with modulus k. The extremal function $f(z)$ maps the annulus $1 < |z| < R$ onto the whole w-plane with two slits along intervals of the real axis: $-\infty < w \leq f(R) < 0$ and $0 \leq w \leq f(-1) < 1$. Thus, for $f(z) \in F$ $(n = 2)$, the sharp estimate holds

$$|f'(z_0)| \leq \frac{4K}{z_0 \ln R \left|\operatorname{sn}\left(\frac{2K}{\ln R} \ln z_0;\, k\right)\right|}.$$

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Note: Figure translations are in progress. See original paper for figures.

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