V. F. Gaposhkin

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Abstract

Full Text

V. F. Gaposhkin

THE LOCALIZATION PRINCIPLE AND SYSTEMS $\{\varphi(nx)\}$

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In the author' s paper ([1]) a class of systems of the form $\{\varphi(nx)\}$, "close" in their properties to the trigonometric system, was singled out. This class is characterized by the following properties:

a)

$$\varphi(x) = \sum_{k=1}^{\infty} b_k \sin kx, \quad \text{where} \quad \sum_{k=1}^{\infty} |b_k| \leq C_2 < \infty;$$

b) the function

$$\Phi(z) = \sum_{k=1}^{\infty} \frac{b_k}{k^z}$$

satisfies the inequalities

$$0 < C_1 \leq |\Phi(z)| \leq C_2 \quad \text{for } \operatorname{Re} z > 0.$$

When conditions a) and b) are fulfilled, the system $\{\varphi(nx)\}_{n=1}^{\infty}$ is a complete minimal system in $L_p'(-\pi, \pi)$, $p \geq 1$ (the space of odd 2π-periodic functions with integrable p-th power), and the adjoint system has the form

$$\psi_n(x) = \sum_{d/n} B_d \sin \frac{n}{d} x$$

(the sum is taken over all integer divisors of the number n), where the numbers B_n are uniquely determined from the equations

$$B_1 b_1 = 1, \qquad \sum_{d/n} B_d b_{\frac{n}{d}} = 0 \quad (n > 1). \tag{1}$$

The Fourier series of a function $f(x) \in L'(-\pi, \pi)$ with respect to the system $\{\varphi(nx)\}$ has the form

$$f(x) \sim \sum_{n=1}^{\infty} \tilde{a}_n \varphi(nx), \tag{2}$$

where

$$\tilde{a}_n = (f, \psi_n) = \sum_{d/n} B_d a_{\frac{n}{d}}; \qquad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx,$$

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin nx. \tag{3}$$

In paper ([1]) it was shown that, when conditions a) and b) are fulfilled, the series (2) and (3) possess many similar properties. For example, they simultaneously converge or diverge in the metric $C'(-\pi,\pi)$; $L'_p(-\pi,\pi)$ $(p \geq 1)$, simultaneously converge absolutely or fail to converge absolutely, etc. (see also papers ([2]), where a narrower class of systems was considered). It may be observed that the theorems of papers ([1,2]) concern properties of the series (2) and (3) depending on the behavior of the odd function $f(x)$ on the whole interval $(0,\pi)$. It is natural to ask whether any theorems on equiconvergence at individual points will hold for the series (2) and (3). It is known that convergence of the series (3) at a point x_0 depends only on the properties of the function $f(x)$ in some neighborhood of this point (Riemann' s localization principle for the trigonometric system). Therefore our question reduces to the following: is the localization principle valid for the systems under consideration? The following Theorems 1 and 1′ give a negative answer to this question.

We shall say, as usual, that for the system $\{\varphi_k(x)\}$ the localization principle is valid for convergence (summability by method A) at a certain point-

point x_0, if from the coincidence of two functions $f_1(x)$ and $f_2(x)$ in some neighborhood of the given point it follows that

$$\lim_{n\to\infty} \{S_n(f_1;x_0) - S_n(f_2;x_0)\} = 0,$$

where $S_n(f_i;x)$ are the partial sums (or A-means) of the Fourier series of the functions $f_i(x)$ $(i = 1,2)$ with respect to the system $\{\varphi_k(x)\}$, $f_i \in L'(-\pi,\pi)$. We shall call systems possessing properties a) and b) T-systems.

Theorem 1. *If for the T-system $\{\varphi(nx)\}$ the localization principle for Abel' s summability method is valid at some point x_0, $x_0 \neq \frac{p}{q}\pi$, then*

$$\varphi(x) = b_1 \sin x, \qquad b_1 \neq 0.$$

Here and below p and q are relatively prime integers, $p < q$, $q > 1$.

From Theorem 1 there follows immediately:

Theorem 1′. *If for the T-system $\{\varphi(nx)\}$ the localization principle for convergence is valid at some point x_0, $x_0 \neq \frac{p}{q}\pi$, then $\varphi(x) = b_1 \sin x$, $b_1 \neq 0$.*

If we now consider the case when $x_0 = \frac{p}{q}\pi$, then the function $\varphi(x)$ in the hypotheses of Theorem 1 (or Theorem 1′) is no longer obliged to coincide (up to a factor) with $\sin x$. Put

$$\gamma_{l,\nu} = \sum_{plk \equiv \nu \pmod q} b_k \operatorname{sign}\left(\sin \frac{plk}{q}\pi\right), \qquad 1 \le \nu \le q-1; \quad l = 1, 2, \ldots$$

Notice that $|\gamma_{l,\nu}| = |\gamma_{sq+l,\nu}|$ for $s \ge 1$.

Theorem 2. *Let $\{\varphi(nx)\}$ be a T-system. In order that, for this system, the localization principle be valid at some point $x_0 = \frac{p}{q}\pi$, it is necessary and sufficient that the relations*

$$B_l \gamma_{l,\nu} = 0, \qquad 1 \le \nu \le q-1; \quad \nu \ne p; \quad l = 1, 2, \ldots \tag{4}$$

be satisfied.

It is also easy to show that, for points x_0 of the form $\frac{p}{q}\pi$, the following modified localization principle is valid:

The convergence of the Fourier series of a function $f(x)$ at the point x_0 with respect to any T-system $\{\varphi(nx)\}$ depends only on the behavior of the function $f(x)$ in neighborhoods of the points $x_0, 2x_0, \ldots, (q-1)x_0$.

Let us make the following additional remarks.

1. For any point $x_0 = \frac{p}{q}\pi$ there exist T-systems $\{\varphi(nx)\}$, $\varphi(x) \ne b_1 \sin x$, for which the localization principle is valid at this point (i.e. conditions (4) are satisfied). As such a system one may take the system generated by the function $\varphi(x) = \sin x - \frac{1}{2}\sin qx$. In this case $B_1 = 1$,

$$B_{q^n} = \frac{1}{2^n}\ (n \ge 1), \qquad B_k = 0\ (k \ne q^n).$$

2. All the results carry over without difficulty to T-systems of the form $\{1, \varphi(nx)\}$ in the case when the function $\varphi(x)$ is even.

3. The results of the present note show that systems $\{\varphi(nx)\}$ close to the system $\{\sin nx\}$ in their "global" properties turn out to be far from it in their local properties. However, the possibility is not excluded that among systems $\{\varphi(nx)\}$ which are not T-systems there may be some for which the localization principle is valid.

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REFERENCES

1. V. F. Gaposhkin, *Matem. sborn.*, **51** (93), 2, 239 (1960).

2. K. F. Mayavko, DAN, **118**, No. 1, 29 (1958).

* Theorem 1 remains valid if one considers the localization principle in the class of bounded functions, and Theorem 1′ in the class of continuous functions.

Note: Figure translations are in progress. See original paper for figures.